Math 252 — Fall 2002
Some comments on bifurcations

Background. This is a slightly modified version of the notes posted on the same subject posted on the original Math 252 web page and borrowed from the UTEP "SOS math" project. Links to these resources are available from the original 1998 page available through the "History" section of this site.

One might reasonably begin by looking up the word *bifurcation* in a dictionary. We quote the Concise Oxford Dictionary, 8th Ed., Copyright 1991 Oxford Univ. Press:

1. a. a division into two branches;
   b. either or both of such branches.
2. the point of such a division.

Finding Bifurcation — an Analytic Approach.. A quick way to look for bifurcation values of the parameter for the one-dimensional differential equation

$$\frac{dy}{dt} = f(y, \mu),$$

where $f(y, \mu)$ is a function for which both f and $\partial f/\partial y$ are continuous, begins by finding the simultaneous solutions of the system

$$f(y, \mu) = 0 \qquad (1A)$$

$$\frac{\partial f}{\partial y} = 0. \qquad (1B)$$

For each critical pair (y_0, μ_0), it is necessary to examine the behavior of the equation $y' = f\big(y(t), \mu\big)$ near the equilibrium point $y = y_0$ and try to determine whether bifurcation in fact occurs.

This procedure is very similar to the procedure which you followed in calc 1, when, in order to maximize a function, you first found critical points, and then studied, for each critical point, if it was a maximization point or not.

The solution of these equations may involve many technical difficulties, but examples in this course should allow all solutions to be found easily.

The justification of this procedure is that when the derivative on the left side of in $(1B)$ is **not** zero, one has, for any parameter near μ_0, some equilibrium near y_0 and of the same type (source or sink) and thus there is no change of behavior at that equilibrium point. The other equation, $(1A)$ just says that y is an equilibrium for this value of μ.

Here is another example, made up to show how solving two equations at the same time may be easier than considering them separately: if

$$f(y, \mu) = \sin y - \mu \cos y,$$

there are many possible equilibria for any given value of μ. (For example, when $\mu = 0$, all integer multiples of π are equilibria.) However, if we consider both equations in (1), we have

$$\sin y - \mu \cos y = 0$$
$$\cos y + \mu \sin y = 0$$

Multiplying the first of these by $\sin y$ and the second by $\cos y$ and adding yields $\sin^2 y + \cos^2 y = 0$. That is, if there are any solutions to this system of equations, then $1 = 0$. Because we believe in the consistency of mathematics, we take this as meaning that the system has no solutions.

In higher dimensional problems, the systematic nature of this approach becomes even more valuable.

Some links. Here are a fewreferences (in web readable form) related to bifurcation:

- lecture notes on chaos and bifurcations (more for discrete systems, but same idea). Has very nice pictures.
- Chaos theory: A brief introduction
- Chaos Demonstrations from Caltech
- The chaos game by Devaney
- Chaos notes
- Analysis and modeling of bipedal gait dynamics

  Abstract: *The main focus of the present investigation is the development of quantitative measures to assess the dynamic stability of human locomotion . . . accommodates the study of the complex dynamics of human locomotion and differences among various individuals . . . Changes in the stability of the biped as a result of bifurcations in the four-dimensional parameter space are investigated.*