LINEAR INHOMOGENEOUS SYSTEMS AND SECOND-ORDER EQUATIONS WITH CONSTANT COEFFICIENTS

1. *Inhomogeneous Linear Systems; The Variation-of-Parameters Formula.*

There is no real difference between the process of giving a “closed-form solution” for the scalar inhomogeneous linear equation $y' = ay + f(t)$ and the same process for a vector inhomogeneous linear equation

$$\frac{d\mathbf{Y}(t)}{dt} = \mathbf{A}\mathbf{Y}(t) + \mathbf{F}(t)\ , \tag{1.1}$$

where the vectors may belong to $\mathbb{R}^n$ or $\mathbb{C}^n$ for any dimension n. There are only a couple of points that need to be remembered: **(a)** vector-valued functions are differentiated coördinatewise, and one also integrates them coördinatewise, so that

$$\text{If } \mathbf{G}(t) = \begin{pmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{pmatrix}, \text{ then } \int_a^b \mathbf{G}(t)\,dt = \begin{pmatrix} \int_a^b g_1(t)\,dt \\ \vdots \\ \int_a^b g_n(t)\,dt \end{pmatrix}. \tag{1.2}$$

Thus in particular the Fundamental Theorem of the Integral Calculus holds for vector-valued functions:

$$\frac{d}{dt}\begin{pmatrix} \int_a^t g_1(u)\,du \\ \vdots \\ \int_a^t g_n(u)\,du \end{pmatrix} = \begin{pmatrix} \frac{d}{dt}\int_a^t g_1(u)\,du \\ \vdots \\ \frac{d}{dt}\int_a^t g_n(u)\,du \end{pmatrix} = \begin{pmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{pmatrix}. \tag{1.3}$$

Also **(b)** if $\mathbf{B}(t)$ is a matrix-valued function and $\mathbf{Y}(t)$ is a vector-valued function, then the product rule holds:

$$\frac{d}{dt}\left[\mathbf{B}(t)\mathbf{Y}(t)\right] = \frac{d\mathbf{B}(t)}{dt}\,\mathbf{Y}(t) + \mathbf{B}(t)\,\frac{d\mathbf{Y}(t)}{dt}\ . \tag{1.4}$$

While a coördinate-free proof of this fact can be given, the easiest thing to do is to write everything out in coördinates: if $\mathbf{B}(t) = (\,b_{ij}(t)\,)$ and $\mathbf{Y}(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}$, then the i-th coördinate of $\mathbf{B}\mathbf{Y}(t)$ is given by $\sum_{j=1}^n b_{ij}(t)\cdot y_j(t)$, and its derivative can be computed term-by-term in the usual way:

$$\frac{d}{dt}\Big[\sum_{j=1}^n b_{ij}(t)\cdot y_j(t)\Big] = \sum_{j=1}^n \frac{db_{ij}(t)}{dt}\cdot y_j(t) + \sum_{j=1}^n b_{ij}(t)\cdot\frac{dy_j(t)}{dt}\ . \tag{1.5}$$

We recognize the r. h. s. of (**1.5**) as the sum of the i-th coördinates of $\mathbf{B}'(t)\mathbf{Y}(t)$ and $\mathbf{B}(t)\mathbf{Y}'(t)$, so this detail is checked.

To solve the system (**1.1**) explicitly, all we now have to do is slavishly to imitate the integrating-factor technique that worked so well for the (single) scalar first-order linear equation $y' = ay + f(t)$, and that only in the case in which a is constant.(1) We start with (**1.1**) above, and to put it in a familiar form shift the “**AY**” term to the l. h. s.:

$$\frac{d\mathbf{Y}(t)}{dt} - \mathbf{A}\mathbf{Y}(t) = \mathbf{F}(t) \tag{1.6}$$

(1) To refresh your understanding of this situation, see pp. 110 ff. of BD&H.

The integrating factor "ought to be" $\exp\left(-\int^t \mathbf{A}(u)\,du\right)$, and since $\mathbf{A}$ is constant this is just $e^{-t\mathbf{A}}$. All we need to know about this exponential to complete the argument is that its derivative is $-\mathbf{A}e^{-t\mathbf{A}}$:

$$\begin{aligned}
\frac{d\mathbf{Y}(t)}{dt} - \mathbf{A}\mathbf{Y}(t) &= \mathbf{F}(t) \\
\frac{d}{du}\left[e^{-u\mathbf{A}}\mathbf{Y}(u)\right] &= e^{-u\mathbf{A}}\left[\frac{d\mathbf{Y}(u)}{du} - \mathbf{A}\mathbf{Y}(u)\right] = e^{-u\mathbf{A}}\mathbf{F}(u) \\
e^{-t\mathbf{A}}\mathbf{Y}(t) - \mathbf{Y}(0) &= \int_0^t \frac{d}{du}\left[e^{-u\mathbf{A}}\mathbf{Y}(u)\right]\,du = \int_0^t e^{-u\mathbf{A}}\mathbf{F}(u)\,du \\
\mathbf{Y}(t) &= e^{t\mathbf{A}}\mathbf{Y}(0) + e^{t\mathbf{A}}\int_0^t e^{-u\mathbf{A}}\mathbf{F}(u)\,du\,. \qquad (\mathbf{1.7})
\end{aligned}$$

It is easy to recognize the first term on the r. h. s. of (**1.7**): it is the solution of the homogeneous equation $\mathbf{Y}' = \mathbf{A}\mathbf{Y}$ that takes the "correct" value $\mathbf{Y}(0)$ at the time $t = 0$. The second term is then a particular solution of the original inhomogeneous equation (**1.1**), namely the one that takes the value $\mathbf{0}$ at time $t = 0$. Since the dummy variable of integration in that second term is u, while t "looks constant to the integral," we may bring the factor $e^{t\mathbf{A}}$ inside the integral sign and use the fact that $e^{t\mathbf{A}}e^{-u\mathbf{A}} = e^{(t-u)\mathbf{A}}$ to write (**1.7**) in the alternate form

$$\mathbf{Y}(t) = e^{t\mathbf{A}}\mathbf{Y}(0) + \int_0^t e^{(t-u)\mathbf{A}}\mathbf{F}(u)\,du \qquad (\mathbf{1.7a})$$

which is actually not as handy as the original (**1.7**) for computational purposes but which has certain advantages for conceptualization, as we shall see in §§2–3 below.

This solution of (**1.1**) is called the **Variation-of-Parameters Formula**;(2) the same name is given to its consequences for higher-order scalar equations. There are versions of it for equations $\mathbf{Y}'(t) = \mathbf{A}(t)\mathbf{Y}(t) + \mathbf{F}(t)$ in which $\mathbf{A}(t)$ need not be constant, but we shall not be concerned with those now. Far in your future, you may see versions of it that concern "boundary-value problems" rather than the "initial-value problems" that we have been considering.

2. *Inhomogeneous Linear Equations of Second Order with Constant Coefficients.*

In this § we want to apply the results of §1 to the second-order linear inhomogeneous *scalar* differential equation

$$\frac{d^2y}{dt^2} + p\,\frac{dy}{dt} + qy = g(t) \qquad (\mathbf{2.1})$$

in which p and q are constants. This is the equation that arises when one writes the $F = ma$ equation for a damped spring driven by an external (time- but not position-dependent) force $f(t)$:

$$ma = m\,\frac{d^2y}{dt^2} = -b\,\frac{dy}{dt} - ky + f(t) \qquad (\mathbf{2.2})$$

and divides by m, so $p = b/m$, $q = k/m$ and $g(t) = f(t)/m$. We turn (**2.1**) into a first-order system in a phase plane with coördinates y and v, so that (**2.1**) becomes the system

$$\begin{aligned}
y' &= v \\
y'' = v' &= -qy - pv + g(t) \qquad (\mathbf{2.3s})
\end{aligned}$$

in scalar form; or in vector form, with $\mathbf{Y} = \begin{pmatrix} y \\ v \end{pmatrix}$ and $\mathbf{G}(t) = \begin{pmatrix} 0 \\ g(t) \end{pmatrix}$, it becomes

$$\frac{d\mathbf{Y}(t)}{dt} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}\mathbf{Y}(t) + \mathbf{G}(t)\,. \qquad (\mathbf{2.3v})$$

(2) An alternate name for it is the "variation-of-constants" formula. We reject this name as illogical: how can constants vary?

We now need to compute $e^{t\mathbf{A}}$, where $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}$, and to treat this in full generality we need to diagonalize $\mathbf{A}$ "in letters" rather than "in numbers." Fortunately, $\mathbf{A}$ has a special form[(3)] that enables us to diagonalize it easily—although a certain amount of special pleading will be necessary in the case where the characteristic polynomial of $\mathbf{A}$ has a double root. In any event, the characteristic polynomial of $\mathbf{A}$ is

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \det \begin{pmatrix} \lambda & -1 \\ q & \lambda + p \end{pmatrix} = \lambda^2 + p\lambda + q \; . \tag{2.4}$$

Because the characteristic polynomial is quadratic, we can express its roots using the quadratic formula. Since we are not working with particular numbers, however, we make the following conventions: set

$$\frac{-p + \sqrt{p^2 - 4q}}{2} = \alpha + \beta \text{ and } \frac{-p - \sqrt{p^2 - 4q}}{2} = \alpha - \beta \; . \tag{2.5}$$

That is, $\alpha = -p/2$, the "$-b/2$" term of the quadratic formula, and $\beta = \dfrac{\sqrt{p^2 - 4q}}{2}$, the "square root of the discriminant" term. This makes sense whether the discriminant is positive (the case of two real roots) or negative (the case of two complex conjugate roots); but in the latter case β is a pure imaginary (a real multiple of i), and in this case we shall set $\beta = i\omega$ where ω is real and positive (and is the number we usually interpret as an angular velocity). The two eigenvalues of $\mathbf{A}$ are then $\lambda_1 = \alpha + \beta$ and $\lambda_2 = \alpha - \beta$.

To find the eigenvector belonging to λ_1, we look at the null space of $\lambda_1\mathbf{I} - \mathbf{A}$, *i.e.*, we solve the system of two linear homogeneous equations

$$\begin{pmatrix} \lambda & -1 \\ q & \lambda + p \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{with } \lambda = \lambda_1. \tag{2.6}$$

Since we know the matrix of coefficients is singular, the solution $\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}$ of the first equation that we can see on inspection will automatically satisfy the second equation; so we can use this as our eigenvector $\mathbf{V}_1$ belonging to λ_1. Exactly the same reasoning produces an eigenvector $\mathbf{V}_2 = \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$ belonging to λ_2. We thus have the matrix $\mathbf{S}$ of eigenvectors and its inverse:

$$\mathbf{S} = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}, \qquad \mathbf{S}^{-1} = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{pmatrix} \; . \tag{2.7}$$

With these available we can explicitly write $e^{t\mathbf{A}}$:

$$e^{t\mathbf{A}} = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \left(\frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{pmatrix} \right) \; . \tag{2.8}$$

Carrying out the indicated matrix multiplications in (**2.8**) gives us the explicit matrix expression

$$e^{t\mathbf{A}} = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t} & e^{\lambda_2 t} - e^{\lambda_1 t} \\ \lambda_1\lambda_2(e^{\lambda_1 t} - e^{\lambda_2 t}) & \lambda_2 e^{\lambda_2 t} - \lambda_1 e^{\lambda_1 t} \end{pmatrix} \tag{2.9}$$

for $e^{t\mathbf{A}}$. We could in principle plug this into (**1.7**) above and grind out a fairly explicit solution of (**2.3v**). However, this is really more than we need or want. Rather than the vector $\begin{pmatrix} y(t) \\ v(t) \end{pmatrix}$ in the phase plane,

(3) It's a "companion matrix." See the remarks in the last section of these notes.

we are really only interested in its first coördinate, the **position function** $y(t)$ that gives a particular solution of (**2.1**). Moreover, the vector driving function $\mathbf{G}(t)$ in this situation has a particularly simple form: $\mathbf{G}(t) = \begin{pmatrix} 0 \\ g(t) \end{pmatrix} = g(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. With $y(0) = 0$ and $v(0) = 0$ so that we look at the particular solution of (**2.3v**) represented by the case

$$\begin{pmatrix} y(t) \\ v(t) \end{pmatrix} = \int_0^t e^{(t-u)\mathbf{A}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} g(u)\, du \tag{2.10}$$

of (**1.7a**), and only at the first coördinate of that, we see that only the upper r. h. corner of $e^{(t-u)\mathbf{A}}$ is involved in the integral expression for $y(t)$: we have

$$y(t) = \int_0^t \left[\frac{e^{\lambda_2 (t-u)} - e^{\lambda_1 (t-u)}}{\lambda_2 - \lambda_1} \right] g(u)\, du\ . \tag{2.11}$$

This formula is the **Variation-of-Parameters solution** of the equation (**1.1**).

While this raw form of (**2.11**) can actually be the one best suited to computation when λ_1 and λ_2 are real, it can be rewritten in a form that makes more sense conceptually. This is particularly helpful when the eigenvalues are conjugate complex numbers, so that $\lambda_1 = \alpha + i\omega$ and $\lambda_2 = \alpha - i\omega$. The denominator of (**2.11**) then becomes the pure-imaginary $\lambda_2 - \lambda_1 = (\alpha - i\omega) - (\alpha + i\omega) = -2i\omega$. Both terms of the numerator contain the real factor $e^{\alpha(t-u)}$, and in fact

$$e^{\lambda_2 (t-u)} - e^{\lambda_1 (t-u)} = e^{\alpha(t-u)} \cdot \left[e^{-i\omega(t-u)} - e^{i\omega(t-u)} \right] . \tag{2.12}$$

Now the Euler formula $e^{i\theta} = \cos\theta + i \sin\theta$ can be run backward to express the circular functions in terms of exponentials of pure imaginaries: if we replace θ by $-\theta$ and watch the cosine sit still but the sine change sign, and then add and subtract respectively the two expressions for $e^{i\theta}$ and $e^{-i\theta}$ that we get, we derive the two relations on the last two set-off lines below:

$$\begin{aligned} e^{i\theta} &= \cos\theta + i \sin\theta \\ e^{-i\theta} &= \cos\theta - i \sin\theta \\ e^{i\theta} + e^{-i\theta} &= 2\cos\theta \qquad \text{or} \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad (2.13) \\ e^{i\theta} - e^{-i\theta} &= 2i\, \sin\theta \qquad \text{or} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}\ . \qquad (2.14) \end{aligned}$$

The factor in the integrand of (**2.11**) that comes from $e^{(t-u)\mathbf{A}}$ can thus be rewritten as

$$\frac{e^{\lambda_2 (t-u)} - e^{\lambda_1 (t-u)}}{\lambda_2 - \lambda_1} = e^{\alpha(t-u)} \frac{e^{-i\omega(t-u)} - e^{i\omega(t-u)}}{-2i\omega} = e^{\alpha(t-u)} \frac{\sin \omega(t-u)}{\omega} \tag{2.15}$$

and (**2.11**) will take the form

$$y(t) = \int_0^t \left[e^{\alpha(t-u)} \frac{\sin \omega(t-u)}{\omega} \right] g(u)\, du\ . \tag{2.16}$$

This function should look familiar: $e^{\alpha t} \sin \omega t$ is a solution of the homogeneous equation $y'' + py' + qy = 0$, the equation of the freely vibrating spring, and so for each fixed u the function $t \mapsto \dfrac{e^{\alpha(t-u)} \sin \omega(t-u)}{\omega}$ is a solution[4] of the equation of motion of the freely vibrating spring, "normalized" to have position 0 and velocity 1 at the time when $t = u$. Thus we are dealing with a function that is natural to the problem; as we

(4) Note that because the homogeneous equation is autonomous, the time-translate of a solution by a time-change u is again a solution.

will observe in §3 below, from a physical point of view the integral form of the solution can be considered as a "superposition" of things that the external driving force has done to the spring at times τ between "the time the clock was started" and the current time t.

Similar plastic surgery can be performed in the case in which the eigenvalues of $\mathbf{A}$ are real: instead of the "circular functions" cosine and sine, we need the **hyperbolic functions**, which are defined directly in terms of the exponential function: $\cosh t = \dfrac{e^t + e^{-t}}{2}$ and $\sinh t = \dfrac{e^t - e^{-t}}{2}$. The resemblance of these definitions to the conclusions of (**2.13–14**) above is obvious. We can thus write in the case where β is real

$$\frac{e^{\lambda_2(t-u)} - e^{\lambda_1(t-u)}}{\lambda_2 - \lambda_1} = e^{\alpha(t-u)}\,\frac{e^{-\beta(t-u)} - e^{\beta(t-u)}}{-2\beta} = e^{\alpha(t-u)}\,\frac{\sinh \beta(t-u)}{\beta} \qquad (\mathbf{2.17})$$

and (**2.11**) will take the form

$$y(t) = \int_0^t \left[e^{\alpha(t-u)}\,\frac{\sinh \beta(t-u)}{\beta} \right] g(u)\,du\,. \qquad (\mathbf{2.18})$$

(In this case the formula has only the purpose of displaying the formal identity of the two cases in which β is real and $\beta = i\omega$ respectively; if one actually has to perform the integrations one rarely wants to involve both exponentials and hyperbolic functions when the exponentials $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ would be easier to work with directly. When β was pure-imaginary and circular functions were involved, it was natural to distinguish those from the factor $e^{\alpha t}$.) Again we observe that the factor enclosed in square brackets in the integrand is a solution of the homogeneous equation, normalized to have position 0 and velocity 1 at the time when $t = u$.

We have been ignoring the case in which the characteristic equation $\lambda^2 + p\lambda + q = 0$ has two equal roots, and in which it is possible neither to diagonalize the matrix $\mathbf{A}$ nor to divide by $\lambda_2 - \lambda_1$. Without attempting to justify the derivation rigorously, however, we can derive this case from the one we just analyzed. In the two-equal-roots case, the root(s) is/are real and equal to α; the radical β is zero. If we take the limit as $\lambda_2 \to \lambda_1$, or equivalently as $\beta \to 0^+$ through real values, we find that by L'Hôpital's rule (taking the limit, and therefore the derivatives, with respect to β) we have

$$\begin{aligned} \lim_{\lambda_2 \to \lambda_1} \frac{e^{\lambda_2(t-u)} - e^{\lambda_1(t-u)}}{\lambda_2 - \lambda_1} &= \lim_{\beta \to 0^+} e^{\alpha(t-u)}\,\frac{\sinh \beta(t-u)}{\beta} \\ &= \lim_{\beta \to 0^+} e^{\alpha(t-u)}\,\frac{(t-u)\cosh \beta(t-u)}{1} = (t-u)e^{\alpha(t-u)}\,. \end{aligned} \qquad (\mathbf{2.19})$$

Not coincidentally, this is the solution $te^{\alpha t}$ of $y'' + py' + qy = 0$ with initial conditions $y(0) = 0$ and $y'(0) = 1$ in the two-equal roots case, evaluated at the "past time" $t - u$. It is easy to check—after the fact—that the function

$$t \mapsto \int_0^t (t-u)e^{\alpha(t-u)} g(u)\,du \qquad (\mathbf{2.20})$$

actually gives the particular solution of $y'' + py' + qy = g(t)$ for which $y(0) = 0$ and $y'(0) = 0$ in the two-equal-roots case. Thus the form of the particular solution remains the same in all three cases, although the formulas appear to differ.

In all three cases, the expressions derived in this § provide a *particular solution* of the inhomogeneous equation. The general solution is then formed by adding the general solution of the associated homogeneous equation. A convenient property of these particular solutions, by the way, is that *they all have initial position and velocity zero.* Thus the initial conditions can be met entirely by adding the solution of the associated homogeneous equation that satisfies them—no "corrections" have to be made because the solution of the inhomogeneous equation contributes initial position or velocity. We saw this in general back in (**1.7a**) above. (Contrast this with what happens with the "guesswork" solution $\dfrac{e^{st}}{s^2 + ps + q}$ of $y'' + py' + qy = e^{st}$.) Of course, getting the initial conditions to come out right is usually easier than computing those integrals, so *for most garden-variety concrete problems involving civilized-looking r. h. sides, the methods employed in §§4.1–4.4 of the text should be employed.* Only for theoretical purposes and for certain driving functions (*e.g.*, discontinuous ones) should the integral formula be preferred.

3. *Remarks.*

(a) In the discussion of the matrix of coefficients $\begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}$ in the early part of §2 above, we mentioned that it was an example of a **companion matrix**. In general, if $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is an n-th degree polynomial, the matrix $\mathbf{A}$ in

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix}; \qquad \mathbf{V}_\lambda = \begin{pmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \end{pmatrix} \tag{3.1}$$

with 1's on the superdiagonal and the negatives of the coefficients of $p(x)$ strung out along its bottom row (or sometimes the transpose of this matrix) is called the **companion matrix of the polynomial**. (Of course in our case $n = 2$ and the matrix does not have as many rows and columns as indicated in (**3.1**)!) It is fairly easy to check (use expansion by minors of an appropriately chosen row or column) that its characteristic polynomial $\det(\lambda\mathbf{I} - \mathbf{A}) = p(\lambda)$; it is just the polynomial we started with (with λ plugged in where the indeterminate x went). Thus the eigenvalues of $\mathbf{A}$ are just the roots of $p(\lambda) = 0$. Moreover, it is easy to check that the (unique[5]) eigenvector of $\mathbf{A}$ belonging to a root λ of $p(\lambda) = 0$ is the vector $\mathbf{V}_\lambda$ of (**3.1**). It was this special form that made our lives so easy when we needed to be able to write "general eigenvectors" for a "general matrix."

(b) We mentioned that the integral solution of $y'' + py' + qy = g(t)$ had a physical interpretation as a "superposition" of motions. The following explanation is not rigorous but is good enough for intuitive classical physics. To make things intuitive we assume that the spring is actually going to vibrate, so there are circular-function "harmonics" in the solution of the equation of motion; but the considerations in the overdamped (hyperbolic-functions) or critically-damped ($te^{\alpha t}$) cases would be exactly the same.

If one integrates the equation $F = ma$ or $F = my''$ over a short interval of time $\Delta\tau$, one finds (since over the short interval the force is approximately constant) that $F\Delta\tau = my''\Delta\tau = m\dfrac{dv}{dt}\Delta\tau = m\Delta v$: the "impulse" $F\Delta\tau$ is the change $m\Delta v = \Delta(mv)$ in the momentum of the moving particle during the time interval of length $\Delta\tau$ (physicists sometimes call this relation the **impulse-momentum theorem**). Now if the motion of the particle is governed by the equation (**2.2**) or equivalently (as dividing by the mass will show) (**2.1**), and if it starts at rest and at the equilibrium position at time 0, then if the only thing that happened to it between time 0 and time t was the application of a force $f(\tau)$ for a very short time interval of length $\Delta\tau$ at approximately time τ, then it would have had a momentum at time τ approximately equal to $\Delta(mv) = f(\tau)\Delta\tau$, and its velocity would thus have been approximately $\Delta v = \dfrac{f(\tau)}{m}\Delta\tau = g(\tau)\Delta t$ at a very very short time after the time τ. (Under the assumption that nothing happened before time τ, the values of both y and $v = y'$ in the equation of motion would have been zero, so they did not contribute to the acceleration y'' and we did not have to include those terms in our application of the impulse-momentum theorem.) After that time no external force would have been applied, so the spring would have vibrated autonomously just as it would have done with initial position 0 and initial velocity Δv at time τ, which means that its position at time t would have been (approximately)

$$e^{\alpha(t-\tau)}\,\frac{\sin\omega(t-\tau)}{\omega}\,\Delta v = e^{\alpha(t-\tau)}\,\frac{\sin\omega(t-\tau)}{\omega}\,g(\tau)\,\Delta\tau\,. \tag{3.2}$$

[5] In the sense that all other eigenvectors belonging to λ are proportional to it.

This is true because $e^{\alpha(t-\tau)}\dfrac{\sin\omega(t-\tau)}{\omega}$ is the solution of the homogeneous equation = autonomous spring equation with position 0 and velocity 1 at time 0, so plugging in the argument $t-\tau$ makes those initial conditions hold at time τ rather than time 0—multiplying by $g(\tau)\Delta\tau$ then makes the velocity equal $g(\tau)\Delta\tau$ as required by the impulse-momentum theorem.

Now all we have to do is to break up the time interval $[0,t]$ into many little pieces by partition-points $0=\tau_0<\tau_1<\cdots<\tau_n=t$. If the length of the j-th interval is $\Delta\tau_j=\tau_j-\tau_{j-1}$, then adding up the cases of (**3.2**) that occur at the various times τ_{j-1} for $j-1=0,\ldots,(n-1)$ gives us

$$\sum_{j=1}^{n} e^{\alpha(t-\tau_{j-1})}\frac{\sin\omega(t-\tau_{j-1})}{\omega}\,g(\tau_{j-1})\,\Delta\tau_j\ . \tag{3.3}$$

This is obviously a Riemann sum: as $\max\limits_{1\le j\le n}\Delta\tau_j\to 0$ the discrete sum in (**3.3**) turns into the integral

$$\int_0^t e^{\alpha(t-\tau)}\frac{\sin\omega(t-\tau)}{\omega}\,g(\tau)\,d\tau \tag{3.4}$$

which (except for the trivial change of the name of the dummy variable of integration from u to τ) is the same expression as (**2.16**) above. Thus the solution of (**2.2**) that we derived purely mathematically can be given a physical interpretation in which the integral appears as the sum(6) or "superposition" of the motions that would have occurred for momentary "impulses" with force $f(\tau)$ at times τ earlier than t. For this reason the function $e^{\alpha(t-\tau)}\dfrac{\sin\omega(t-\tau)}{\omega}$ is frequently called the **impulse function** (sometimes **unit impulse function**, and sometimes also **with impulse at** τ) for the equation (**2.1**).

It goes without saying that all the considerations above apply, *mutatis mutandis*, to the cases in which the eigenvalues are real, including the case of a single eigenvalue of multiplicity 2.

(c) Some of you will eventually deal quite frequently with integrals of the form(7)

$$\int_0^t k(t-u)g(u)\,du \tag{3.5}$$

or more likely, you will have a "fixed function" $k(t)$ and "variable functions" $g(t)$ and will consider the (linear!) mapping

$$g(t)\mapsto\int_0^t k(t-u)g(u)\,du \tag{3.6}$$

that transforms g into a new function. The correspondence between the driving function $g(t)$ and the particular solution of (**2.1**) given by (**2.16**) is evidently of this form. Such integrals are called **convolutions**;(8) the integral in (**3.5**) is called the **convolution** of the functions k and g. In situations like (**3.6**), where $k(\cdot)$ is thought of as "fixed" but $g(\cdot)$ is thought of as a "variable function," the operation that (**3.5**) defines is called a **convolution transform** and k is called a **convolution kernel**. Most of you will not have to remember this verbiage, but if you need it you will have seen it in the first context in which it usually appears.

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(6) The reason that one can take a sum here is that any partial sum involving times τ_k for $k<j-1$ will be a solution of the homogeneous (autonomous, un-driven) equation of motion, and thus those terms account for the position of the particle at time τ_{j-1}; one then only has to add the increment corresponding to the impulse at time τ_{j-1} to get the current position of the particle at time τ_j. One can thus add the terms, in the order of their times, to produce the Riemann sum while preserving the physical interpretation.

(7) For example, these are found in the beginning probability course, Math 477, where they occur when one computes the probability density of the sum of two independent nonnegative random variables whose respective probability densities are known.

(8) The verb that goes with this noun is to *convolve*, not "to convolute." Mr Beals, our august Dean of Educational Innovation (or some such thing) manifests signs of considerable discomfiture when people invent that other verb in his presence.