Harmonic Analysis Notes

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Lecture 1

Review

Fourier Inversion

For $f \in L^1(\mathbb{R}^n)$, we define the Fourier transform $\hat{f} : \mathbb{R}^n \to \mathbb{R}$ by:

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx$$

where $\xi \in \mathbb{R}^n$, and $\cdot$ is the dot product. $\hat{f}$ can be seen to be bounded and continuous without much difficulty[1]. We note that this transform is sometimes normalised differently. A natural question is, can we reverse this transform? If we also have $\hat{f} \in L^1$, the answer is yes:

Proposition 1 (Fourier Inversion Formula)**.** *If $f \in L^1(\mathbb{R}^n)$, and further $\hat{f} \in L^1(\mathbb{R}^n)$, then:*

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi \tag{1}$$

for almost every $x \in \mathbb{R}^n$.

Why might this be true?
Decomposing the $e^{2\pi i x \cdot \xi}$ term, for varying ξ we can imagine these as travelling waves in direction $\frac{\xi}{|\xi|}$ with speed ξ, unless $\xi = 0$, which will just give the indicator function of $\mathbb{R}^n$. When we did our initial Fourier transform, we essentially used f to modulate

[1]Pass the absolute value into integrand for the first, use the Dominated Convergence Theorem for the second

'combining' these waves. That is, we took a superposition of these waves by integrating (so our solution does not blow up), and used f to determine the amplitude of each wave in this superposition .

If we imagine $\hat{f}$ as a sum of of these waves, then in the inversion formula the new factor in the integrand will uniquely superimpose with each of these waves to form a new wave. The wave modulated by $f(x)$ will superimpose to form the indicator function of the space. If we integrate, we might guess heuristically that all other terms in the 'sum' vanish by orthogonality, leaving $f(x)$ (considering the case in $\mathbb{R}$ with 1-periodic functions and $\{e^{2\pi inx}\}_{n\in\mathbb{Z}}$ makes this idea more concrete). It will occur that using all of the plane waves for such transforms leads to issues, and so sometimes we might consider only using 'special' sub-collections of the plane waves, leading later to the study of Fourier restriction theory.

We now provide a sketch proof:

Proof. Let $f, g \in L^1(\mathbb{R}^n)$. By using the Fubini/Tonelli theorem to integrate the function $F(\xi, y) = g(\xi)f(y)e^{-2\pi i\xi\cdot y}$ in two different ways, leading to the formula:

$$\int_{\mathbb{R}^n} g(\xi)\hat{f}(\xi)d\xi = \int_{\mathbb{R}^n} \hat{g}(y)f(y)dy \tag{2}$$

This is close to what we want, as if we were able to take $g(\xi) = e^{2\pi i\xi\cdot x}$ for fixed x we would have one side of the inversion formula, but this g is not in L^1. Can we approximate it by functions that are, and choose these functions so that they have Fourier transform making the right hand side an approximation to the identity of f? That is, we might want these transforms to approximate to a good kernel.

Consider the functions $g_\delta(\xi) = e^{-\pi\delta|\xi|^2}e^{2\pi i\xi\cdot x}$, where $0 < \delta < 1$. It can be seen that as $\delta \to 0$ we have both desired conditions. If we substitute g_δ into the inversion formula and take the limit of each side as $\delta \to 0$ the fact that $\hat{f} \in L^1(\mathbb{R}^n)$ allows us to apply the dominated convergence theorem on the left hand side, and on the right hand side we may use properties of good kernel limits. Evaluation of these techniques leads to the result. ☐

Remark. It is not possible to drop the condition $\hat{f} \in L^1$: from a heuristic standpoint the Dominated Convergence Theorem may be violated, leading to a different limit on the left hand side of (2), but no change to the right hand side. What is possibly not immediately clear is that such an $\hat{f}$ should exist given that $f \in L^1$. Considering $f = \chi_{[-1,1]^n}$, we see that if $\hat{f} \in L^1(\mathbb{R}^n)$ then conditions of (1) are satisfied and so by the DCT f is itself continuous almost everywhere, a contradiction.

One can also see easily by using the dominated convergence theorem that:

$$\left|\left|\hat{f}\right|\right|_{L^\infty} \leq ||f||_{L^1}$$

Schwartz Functions

Definition 1. We call $f \in \mathscr{S}(\mathbb{R}^n)$ a **Schwartz function** if $f \in C^\infty(\mathbb{R}^n)$ and it satisfies a decay condition. Namely, such that for any $\alpha, N \in \mathbb{N}$, there exists $C_{\alpha,N}$ such that:

$$|\partial^\alpha f(x)| \leq \frac{C_{\alpha,N}}{(1+|x|)^N}$$

for all $x \in \mathbb{R}^n$.

Our primary interest in these functions comes from the fact that they are well-behaved when integrating over $\mathbb{R}^n$ due to the decay condition, and they are dense in $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$.

Example 1. Smooth functions of compact support, which are important as an easier to classify dense subset in L^p space for $1 \leq p < \infty$.

Example 2. The Gaussian $e^{-\pi|x|^2}$.

Extending Fourier Inversion

What can we say about Fourier inversion for f in other $L^p(\mathbb{R}^n)$ spaces? Since f is not necessarily integrable itself, this is somewhat ill-formulated - how do we define the Fourier transform for functions in these spaces? If $p = 2$, we can do the following:

Definition 2. For $f \in L^2(\mathbb{R}^n)$, take $f_j \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ such that $f_j \to f$ in the L^2 norm. Then the Fourier transform $\hat{f}$ of f is:

$$\hat{f} = \lim_{j\to\infty} \hat{f}_j$$

Such f_j do exist, for example, by considering a set that is dense in both spaces (such as the simple functions). [well-defined, unique].
In this case, we have Plancherel's theorem:

Theorem 1. *For all $f \in L^1 \cap L^2$, we have that $\hat{f} \in L^2$, with:*

$$||f||_{L^2} = \left|\left|\hat{f}\right|\right|_{L^2}$$

by density, this extends to all $f \in L^2$.

Proof. In the case that f is Schwartz, we define □

In particular, if we define operators $S_R : L^2 \to L^2$ for $R \in \mathbb{R}^+$ by:

$$S_R(f) = \int_{|\xi| \le R} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$$

then we have that:

$$S_R(f) \to_{L^2} f$$

as $R \to \infty$, which is our analogue of Fourier inversion. Since we have now defined the Fourier transform in both L^1, L^2, we can define it for f which decompose into functions of L^1 and L^2. That is, it can be defined for $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$ by applying the Fourier transform to the two elements in the decomposition. We can extend this further by using the following:

Theorem 2 (Riesz-Thorin Interpolation)**.** *Suppose T is a bounded linear operator both as a map $T : L^{p_1} \to L^{q_1}$ and $T : L^{p_2} \to L^{q_2}$. Then T extends to a bounded linear operator $T : L^p \to L^q$ whenever there exists $\theta \in (0, 1)$ such that $(1/p, 1/q) = (1 - \theta)(1/p_1, 1/q_1) + \theta(1/p_2, 1/q_2)$. Further, we have the bound:*

$$||T||_{L^p \to L^q} \le ||T||_{L^{p_1} \to L^{q_1}}^{1-\theta} ||T||_{L^{p_2} \to L^{q_2}}^{\theta}$$

Proof. □

much as in the $p = 2$ case, we might hope again that f arises as some sort of 'Fourier inversion limit'. To this end, we can define the operators S_R as previously for any $f \in L^p$ for $1 \le p \le 2$ by using Theorem 2. Recall this was:

$$S_R f = \int_{|\xi| \le R} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$$

So we might ask:

Q 1. Does $S_R(f) \to f$ in the L^p norm as $R \to \infty$?

It turns out this is related to the following question:

Q 2. Is it true that for all Schwartz functions $f \in \mathscr{S}(\mathbb{R}^n)$ that $||S_1(f)||_{L^p} \lesssim_{p,n} ||f||_{L^p}$?

where we recall that if $A \lesssim_{\{a_i : i \in I\}} B$, then $A \le CB$ where C is a function of the a_i, that is, is constant otherwise. For example, the above says that $||S_1(f)||_{L^p} \le C(p, n) ||f||_{L^p}$ for some function $C(p, n)$ of the degree p of the L^p space, and the dimension n.

The Inversion Questions

A positive answer to Q2 will also give a positive answer to Q1; see reasoning below. Harmonic analysts usually think of Q2 as a quantitative version of the qualitative question Q1, and prefer to analyze the quantitative question over the qualitative one. In the analogous context of summability of Fourier series, the qualitative and quantitative versions are actually known to be equivalent, via the uniform boundedness principle. We show below why a positive answer to Q2 implies a positive answer to Q1.

Sketch proof. The condition in Q2 implies that there exists $c \in \mathbb{R}^+$ such that $||S_R f||_{L^p} \leq c\,||f||_{L^p}$ for all Schwartz functions f and all $R > 0$. Since the Schwartz functions are dense in all the relevant L^p spaces, for any general $f \in L^p$ we may choose f_j Schwartz converging to f in the L^p norm. As a result:

$$\begin{aligned} ||S_R f - f|| &\leq ||S_R f - S_R f_j|| + ||S_R f_j - f_j|| + ||f_j - f|| \\ &\leq (1 + c)\,||f_j - f|| + ||S_R f_j - f_j|| \end{aligned}$$

where all the norms are L^p norms. The first term can be reduced fairly easily, but the second is problematic for $p = 1$[2]. To avoid the problem, we want $\hat{f}_j$ to have compact support. So instead first choose g_j as Schwartz functions that converge to f in the L^p norm. Now take a Schwartz function ϕ so that $\hat{\phi}$ is compactly supported in the unit ball[3], and consider $\hat{g}_j\hat{\phi}$. These are now compactly supported, but we don't have our original convergence behaviour any longer.

So now for $\epsilon > 0$ define $\phi_\epsilon(x) = \epsilon^{-n}\phi(\epsilon^{-1}x)$, and see that $\widehat{\hat{g}_j\hat{\phi}} = g_j * \phi_\epsilon \to g_j$ in the L^p norm as $\epsilon \to 0^+$[4] As a result, we can choose ϵ_j so that this is arbitrarily close, and let $f_j = g_j * \phi_{\epsilon_j}$ be our sequence. In particular, $||S_R f_j - f_j|| \to 0$ as $R \to \infty$. If we then choose j such that $||f_j - f|| < \frac{\epsilon}{1+c}$, and choose the other term less than ϵ, we get our bound.

With this implication, we focus our attention on Q2.

[2] One may try bounding the relevant integral by using the Fourier inversion formula and integration by parts.

[3] One might use inversion on a bump function supported on the unit ball to obtain such a function.

[4] What we have effectively done is concentrate the support of ϕ towards origin and re-normalised it to behave like ϕ, which gives us something like a Dirac point mass or good kernel.

The Answer in 1 Dimension

Fix $p \in [1,2)$. We start by analysing the question when $n = 1$, that is, by considering:

$$S_1 f = \int_{-1}^{1} \hat{f}(\xi) e^{2\pi i x \xi} d\xi$$

We write this as a convolution with a kernel K [convolution recall here?], such that $S_1 f = f * K$. This K is given by:

$$\begin{aligned} K(y) &= \int_{-1}^{1} e^{2\pi i y \xi} d\xi \\ &= \frac{\sin(2\pi y)}{\pi y} \end{aligned}$$

This is 'almost in $L^1(\mathbb{R})$', but isn't[5]. It does, however, decay strongly enough at infinity for it to be in L^p for $p > 1$. If we were to imagine, for sake of heuristics, that $K \in L^1$, we could obtain that:

$$\begin{aligned} ||S_1 f||_{L^p} &= ||f * K||_{L^p} \\ &\leq ||f||_{L^p} ||K||_{L^1} \end{aligned}$$

and we would have the result. The second line is a special case of Young's convolution inequality, and this case can be proved using Theorem 2 with T the convolution operator by K. Namely, considering that this T is bounded as an operator from both $L^1 \to L^1$ (By using Fubini and translation invariance) and as one from $L^\infty \to L^\infty$ (by bounding the kernel in the integral above by $||K||_{L^\infty} \leq ||K||_{L^1}$), application gives the desired result (relaxing the exponents by forcing the relevant operator norms to be greater than 1 by re-scaling first).

This does not really hold, but it almost gives our desired result. In fact by using singular integrals and a Hilbert transform it is possible to show that, for $p > 1$, we have:

$$||S_1 f||_{L^p(\mathbb{R})} \lesssim ||f||_{L^p(\mathbb{R})}$$

[5]Consider that for $y \in [k+1/8, k+3/8]$ where $k \in \mathbb{Z}$, the numerator has absolute value bounded below. Integrating over only these sets the integral is itself bounded below by the integral of $\frac{C}{|y|}$ over these sets, which is infinite. Thus the integral over the whole space is also infinite

So by our earlier analysis we have that $S_R f \to f$ in the L^p norm as $R \to \infty$. If $p = 1$, then this convergence might not occur. In fact, S_1 is unbounded on L^1, as one can see by considering a sequence of L^1 functions converging to a δ function in the sense of distributions. This completes our cases of interest for $n = 1$.

The Answer in Higher Dimensions

For $n = 2$, the answer to these questions is understood (see work of Carleson-Sjolin, and related work of Stein and Fefferman). For $n \geq 3$, there are partial results but also plenty of open questions. In higher dimensions, we might hope to utilise some of the new geometry/structure of the space to answer the question (such as curvature). For example, what if we took our integral over boxes instead of balls? i.e. considered the operator:

$$S'_R f = \int_{[-R,R]^n} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$$

Somewhat unfortunately, this is drastically different. The box structure allows us to 'reduce the dimension' of this problem, and it essentially becomes the problem of the previous section over 1 dimension, where the problem is well understood. So what can we do? We might hope to use our same kernel idea again for Schwartz functions f, so we write $S_1 f = f * K$ where:

$$K(x) = \int_{\overline{B}(0,1)} e^{2\pi i x \cdot \xi} d\xi$$

where this integral is over the closed ball of radius 1 about 0. It has been shown that $K(x)$ satisfies:

$$|K(x)| \lesssim (1 + |x|)^{-\frac{n+1}{2}}$$

for all $x \in \mathbb{R}^n$ (see e.g. computation in Stein's harmonic analysis book). In fact, we have the asymptotic:

$$K(x) = \sum_{\pm} C_{\pm} \frac{e^{\pm 2\pi i |x|}}{|x|^{\frac{n+1}{2}}} + \epsilon(x) \tag{3}$$

where $\epsilon(x)$ is an error term, that is, a term smaller than the main term in some sense. This was done using 'stationary phases', a technique that considers the oscillatory nature and cancellation thereof. Using this asymptotic, Herz (in 1954) showed that

if $1 \leq p \leq \frac{2n}{n+1}$, then $||S_1 f||_{L^p} \not\lesssim ||f||_{L^p}$, that is, we do not have the kind of inversion we want. He did so by testing with[6] $f(x) = (1-|x|^2)^{\alpha}_{+}$, where $\alpha < \frac{n+1}{2n}$ (in fact, it would have been simpler to simply test against the characteristic function of the unit ball). Considering the limit of $\frac{2n}{n+1}$ as $n \to \infty$, namely 2, we see that this alone guarantees we almost always have failure in high dimensions.
What about when we take p such that $\frac{2n}{n+1} < p < 2$? Fefferman (in 1971), showed with an argument utilising a probabilistic argument and ideas from geometric measure theory (Kakeya sets) that it *does not* hold for this range of values. This is now called the Ball Multiplier Theorem, the name taking inspiration from the fact that:

$$\widehat{S_1 f} = 1_{|\xi| \leq 1}$$

the indicator function of the closed unit ball. This completes our study of these cases: if $n \geq 2$, and $p \neq 2$, generally f is NOT the L^p limit of $S_R f$ as $R \to \infty$.

A New Question: Bochner-Riesz summability

How do we solve this issue? we still want something like Fourier inversion, but our previous techniques just don't work. Well, why don't they work? A central issue in the analysis of $S_1 f$ is found by re-writing it as an integral over the whole space, but now with the indicator function of the ball in the integrand. The specific problem is that the indicator function on the ball is discontinuous: we have a jump on the boundary from 1 to 0. What happens if we consider something smoother to have in the integrand? Consider:

$$S_R^{\alpha} f = \int_{\mathbb{R}^n} \left(1 - \frac{|\xi|^2}{R^2}\right)_+^{\alpha} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$$

in the case $\alpha = 0$, we fall back to the ball multiplier case, but what about when $\alpha > 0$? We consider a question analogous to Q2:

Q 3. For fixed $n \geq 2$, for which α, p do we have:

$$||S_1^{\alpha} f||_{L^p} \lesssim ||f||_{L^p}$$

when f is a Schwartz function?

if Q3 considered as a condition holds for certain $p \in (1,2)$ with a given α, we have similarly to the ball multiplier case that $S_R^{\alpha} f \to f$ in the L^p norm for all $f \in L^p$. We again try our kernel technique, writing $S_1^{\alpha} f = f * K_{\alpha}$ where:

[6] Recall that the + in the subscript denotes the positive part of the function

$$K_\alpha(x) = \int_{\overline{B}(0,1)} (1-|\xi|^2)^\alpha e^{2\pi i x\cdot\xi} d\xi$$

a similar asymptotic to (3) can be obtained:

$$K_\alpha(x) \sim C \frac{e^{2\pi i|x|}}{|x|^{\frac{n+1}{2}+\alpha}}$$

which gives rise to a modification of Herz's argument, such that the condition version of Q3 fails if $1 \leq p \leq \frac{2n}{n+1+2\alpha}$ (alternatively, just plug in $f =$ characteristic function of a unit ball, to see that S_1^α is unbounded on $L^p(\mathbb{R}^n)$ if $1 \leq p \leq \frac{2n}{n+1+2\alpha}$). But what about the other values? Do they fail like in the ball multiplier case? This is the subject of the Bochner-Riesz conjecture:

Conjecture 1 (Bochner-Riesz)**.** *If $n \geq 2$, and we have that:*

$$\frac{2n}{n+1+2\alpha} < p < \frac{2n}{n-1-2\alpha},\ 0 < \alpha < \frac{n-1}{2}$$

then:

$$||S_1^\alpha f||_{L^p} \lesssim ||f||_{L^p}$$

Equivalently:

Conjecture 2. *Let $\varphi_\delta(\xi)$ be a smooth function supported on a δ-neighbourhood of the sphere S^{n-1} normalised so that it is 1 near $|\xi| = 1$. Then if for all $\delta > 0$ we have:*

$$\left|\left|\varphi_\delta(\xi)\hat{f}(\xi)e^{2\pi i x\cdot\xi}\right|\right|_{L^p} \lesssim \delta^{-((n-1)(1/2-1/p)-1/p)} ||f||_{L^p}$$

for all $p > \frac{2n}{n-1}$, we have that:

$$||S_1^\alpha f||_{L^p} \lesssim ||f||_{L^p}$$

The relevance being that the multiplier in S_R^α is an attempt to see what happens if you behave smoothly near the sphere. In 1999, Terence Tao proved that the Bochner-Riesz conjecture implies the restriction conjecture, the statement of which is to be covered.