

## HARMONIC ANALYSIS: LECTURE TWO

ASSOCIATE PROFESSOR PO LAM YUNG, TRANSCRIBED BY DOMINIC CONNORS  
(U7289061)

### A HEURISTIC FOR THE FOURIER TRANSFORM

In the previous lecture, we introduced the Fourier transform and the Fourier inversion on  $L^1(\mathbb{R}^n)$ . We extended the inversion formula from  $L^1(\mathbb{R}^n)$  to  $L^2(\mathbb{R}^n)$  using sets dense in both spaces, such as the Schwartz functions  $\mathcal{S}(\mathbb{R}^n)$ , and Plancherel's theorem. We then explored whether the inversion formula could be extended to  $L^p(\mathbb{R}^n)$  for other  $p$  over various dimensions  $n$  with the Riesz-Thorin Interpolation Theorem. This led to a discussion of the Ball Multiplier Theorem and the Bochner-Riesz Conjecture.

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x \cdot \xi} dx \quad f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi)e^{2\pi i x \cdot \xi} d\xi \quad (1)$$

We would like to develop an intuition for the action of the Fourier transform on  $L^p(\mathbb{R}^n)$ . To do this, we will develop an insightful but fundamentally incorrect heuristic for what the Fourier transformation does to indicator functions of  $n$ -dimensional rectangles and use this to evaluate the Fourier transform of functions with compact support in  $L^p(\mathbb{R}^n)$ . Heuristic logical steps are clearly labeled as such hereafter.

### IN ONE DIMENSION

We will firstly consider the one-dimensional case. The Fourier transformation of the Gaussian function given by  $f(x) = e^{-\pi x^2}$  is  $\widehat{f}(\xi) = e^{-\pi \xi^2}$ , the original function. Consider the indicator function  $\mathbb{1}_{[0,1]}$  and its Fourier transformation given by

$$\widehat{\mathbb{1}_{[0,1]}}(\xi) = \int_{\mathbb{R}} \mathbb{1}_{[0,1]}(x)e^{-2\pi i x \xi} dx = \frac{e^{2\pi i \xi} - 1}{-2\pi i \xi} = \frac{e^{-\pi i \xi} \sin 2\pi \xi}{2\pi \xi} \stackrel{\text{Heuristic}}{=} \mathbb{1}_{[0,1]}(\xi) \quad (2)$$

Which decays like  $\frac{1}{\xi}$  at  $\infty$  and is approximately 1 about 0 i.e. the Fourier transformation has some similar properties to the original function. Heuristically, we will think of the Fourier transformation of  $\mathbb{1}_{[0,1]}$  as being itself like the Gaussian above. As you can see below, it does not look much like its Fourier transform but as we will see, it has many of the properties we would expect if it was.

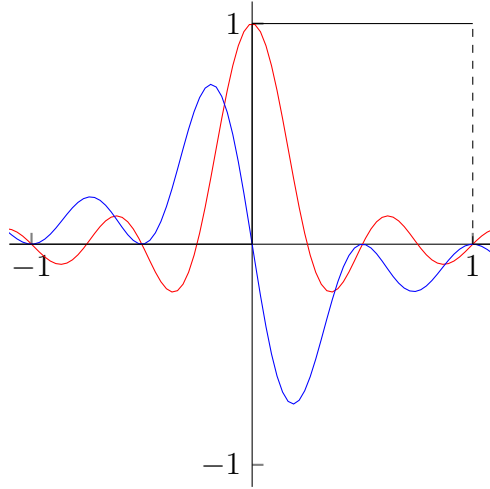


FIGURE 1. The indicator function  $\mathbb{1}_{[0,1]}$  (black) compared with the real (red) and imaginary (blue) parts of its Fourier transform.

We will now generalise our heuristic to indicator functions of general intervals  $I \subset \mathbb{R}$ . Firstly, we will consider the indicator function of a closed interval with an end point at the origin,  $\mathbb{1}_{[0,r]}$  with  $r \in \mathbb{R}$ , and its Fourier transformation given by

$$\begin{aligned} \widehat{\mathbb{1}_{[0,r]}}(\xi) &= \int_{\mathbb{R}} \mathbb{1}_{[0,r]}(x) e^{-2\pi i x \xi} dx = \int_{\mathbb{R}} r \mathbb{1}_{[0,1]}(x) e^{-2\pi i x (r\xi)} dx \\ &= r \widehat{\mathbb{1}_{[0,1]}}(r\xi) \stackrel{\text{Heuristic}}{=} r \mathbb{1}_{[0,1]}(r\xi) = r \mathbb{1}_{[0, \frac{1}{r}]}(\xi) \end{aligned} \quad (3)$$

Where we used the heuristic of (2). Recall that the Fourier transform maps  $L^1(\mathbb{R})$  into  $L^\infty(\mathbb{R})$  and that the Fourier transformation of a function  $f \in L^2(\mathbb{R})$  satisfies the Parseval–Plancherel identity:  $\|\widehat{f}\|_2 = \|f\|_2$ . Our heuristic Fourier transform satisfies the equality  $\|\widehat{\mathbb{1}_{[0,r]}}\|_2 \stackrel{\text{Heuristic}}{=} \|\mathbb{1}_{[0,r]}\|_2 = \sqrt{r}$  and  $\|\widehat{\mathbb{1}_{[0,r]}}\|_\infty = 1 < \infty$  with  $\|\mathbb{1}_{[0,r]}\|_1 = r < \infty$ . So, it acts at least somewhat like the real thing. Qualitatively, we see that concentrated functions are mapped to spread out functions and vice versa in Figure 2, as expected. Note that the heuristic Fourier transform of the heuristic Fourier transform of  $\mathbb{1}_{[0,r]}$  is itself.

Now, we will consider the indicator function of an arbitrary closed interval,  $\mathbb{1}_{[a,a+r]}$  with  $a, r \in \mathbb{R}$ , and its Fourier transformation given by

$$\begin{aligned} \widehat{\mathbb{1}_{[a,a+r]}}(\xi) &= \int_{\mathbb{R}} \mathbb{1}_{[a,a+r]}(x) e^{-2\pi i x \xi} dx = \int_{\mathbb{R}} \mathbb{1}_{[0,r]}(x) e^{-2\pi i (x+a)\xi} dx \\ &= e^{-2\pi i a \xi} \int_{\mathbb{R}} \mathbb{1}_{[0,r]}(x) e^{-2\pi i x \xi} dx = e^{-2\pi i a \xi} \widehat{\mathbb{1}_{[0,r]}}(\xi) \stackrel{\text{Heuristic}}{=} e^{-2\pi i a \xi} r \mathbb{1}_{[0, \frac{1}{r}]}(\xi) \end{aligned} \quad (4)$$

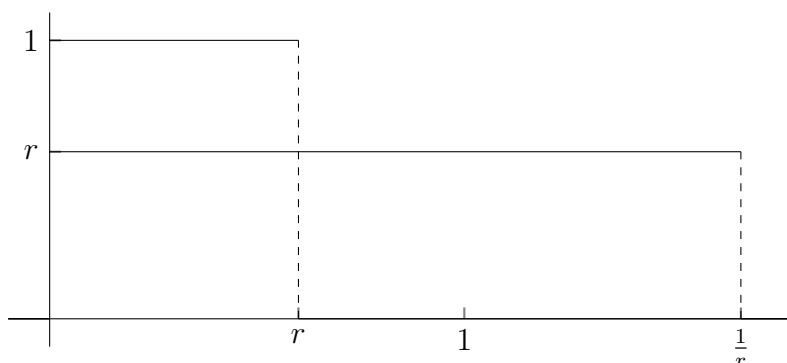


FIGURE 2. The indicator function  $\mathbb{1}_{[0,r]}$  and its heuristic Fourier transform.

Where we used the heuristic of (3). The heuristic Fourier transformation also has the properties of the Fourier transformation described above.

We will slightly loosen our heuristic and say that for an interval  $I \subset \mathbb{R}$ ,  $\widehat{\mathbb{1}}_I(\xi) \stackrel{\text{Heuristic}}{=} e^{-2\pi ia\xi} |I| \mathbb{1}_{[0, \frac{1}{|I|}]}(\xi)$  for any  $a \in I$  (not just an endpoint) and note furthermore that its modulus  $|\widehat{\mathbb{1}}_I(\xi)| \stackrel{\text{Heuristic}}{=} |I| \mathbb{1}_{[0, \frac{1}{|I|}]}(\xi)$ .

### IN HIGHER DIMENSIONS

We will now generalise our heuristic for the Fourier transform of indicator functions of intervals in  $\mathbb{R}$  to that of  $n$ -dimensional rectangles in  $\mathbb{R}^n$ , referred to simply as rectangles hereafter. Consider a rectangle  $\theta \subset \mathbb{R}^n$  with side lengths  $(r_i)_n$ . Let its dual  $\theta^*$  be the rectangle with the same orientation but inverted side lengths  $(\frac{1}{r_i})_n$  centred at the origin. Given a rectangle  $\theta$ , we will denote the tiling of  $\mathbb{R}^n$  by  $\theta^*$  as  $\Pi_\theta$ . Note that we can always choose a basis for  $\mathbb{R}^n$  such that a rectangle is a product of intervals.

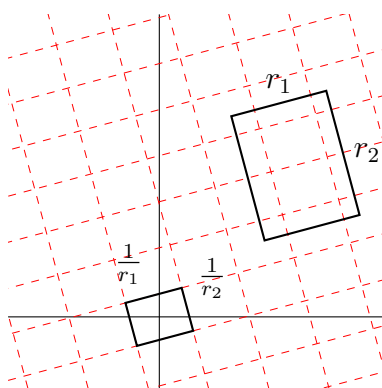


FIGURE 3.  $\theta$ ,  $\theta^*$  and  $\Pi_\theta$

Firstly, we will consider the indicator function of a rectangle in  $\mathbb{R}^2$ ,  $\mathbb{1}_{I_1 \times I_2}$ , and its Fourier transform given by

$$\begin{aligned}
\widehat{\mathbb{1}_{I_1 \times I_2}}(\xi) &= \int_{\mathbb{R}^2} \mathbb{1}_{I_1 \times I_2}(x) e^{-2\pi i x \cdot \xi} dx \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{I_1}(x_1) \mathbb{1}_{I_2}(x_2) e^{-2\pi i x_1 \xi_1} e^{-2\pi i x_2 \xi_2} dx_1 dx_2 \\
&= \left( \int_{\mathbb{R}} \mathbb{1}_{I_1}(x_1) e^{-2\pi i x_1 \xi_1} dx_1 \right) \left( \int_{\mathbb{R}} \mathbb{1}_{I_2}(x_2) e^{-2\pi i x_2 \xi_2} dx_2 \right) \\
&= \widehat{\mathbb{1}_{I_1}}(\xi_1) \widehat{\mathbb{1}_{I_2}}(\xi_2).
\end{aligned} \tag{5}$$

Where Fubini's theorem allowed us to separate the integrals. Hence, we can deduce that for any rectangle  $\theta = I_1 \times \cdots \times I_n$ ,  $\widehat{\mathbb{1}_\theta}(\xi) = \prod \widehat{\mathbb{1}_{I_i}}(\xi_i)$  by an inductive argument. We can now use our heuristic for the Fourier transform of the indicator function of an interval in  $\mathbb{R}$ . For some  $a \in \theta$

$$\begin{aligned}
\widehat{\mathbb{1}_\theta} &= \prod_{i \leq n} \widehat{\mathbb{1}_{I_i}}(\xi_i) \stackrel{\text{Heuristic}}{=} \prod_{i \leq n} e^{-2\pi i a_i \xi_i} |I_i| \mathbb{1}_{[0, \frac{1}{|I_i|}]}(\xi_i) \\
&= \prod_{i \leq n} e^{-2\pi i a_i \xi_i} \prod_{i \leq n} |I_i| \prod_{i \leq n} \mathbb{1}_{[0, \frac{1}{|I_i|}]}(\xi_i) = e^{-2\pi i a \cdot \xi} |\theta| \prod_{i \leq n} \mathbb{1}_{[0, \frac{1}{|I_i|}]}(\xi_i) \\
&\stackrel{\text{Heuristic}}{=} e^{-2\pi i a \cdot \xi} |\theta| \mathbb{1}_{\theta^*}.
\end{aligned} \tag{6}$$

The last equality is not quite true; the indicator function described is that of  $\theta^*$  translated so that a corner lies at the origin, not quite  $\theta^*$  itself. Note also that  $|\widehat{\mathbb{1}_\theta}| \stackrel{\text{Heuristic}}{=} |\theta| \mathbb{1}_{\theta^*}$ .

#### FOURIER TRANSFORMS OF COMPACTLY SUPPORTED FUNCTIONS

Now, we will develop a heuristic understanding as to how the Fourier transform acts on compactly supported functions. Suppose that  $f \in \mathcal{C}_c(\mathbb{R}^n)$ . If  $f$  were smooth as well as compactly supported, it would be a Schwartz function i.e.  $f \in \mathcal{S}(\mathbb{R}^n)$ . Choose a rectangle  $\theta$  that contains 0 and the support of  $f$  so that  $f = f \cdot \mathbb{1}_\theta$ . By way of the convolution theorem and our heuristic

$$\widehat{f}(\xi) = \widehat{f \cdot \mathbb{1}_\theta}(\xi) = \widehat{f}(\xi) * \widehat{\mathbb{1}_\theta}(\xi) \stackrel{\text{Heuristic}}{=} |\theta| \widehat{f} * \mathbb{1}_{\theta^*} = \underset{\theta^* + \xi}{f} \widehat{f} \tag{7}$$

We can heuristically think of the Fourier transform at a point as the average of itself over the tile that contains that point. Hence,  $\widehat{f}$  is constant on each tile in  $\Pi_\theta$ .

Now suppose that we choose a rectangle  $\theta'$  that contains the support of  $f$  but not  $0$ . Choose an  $\omega \in \theta'$  and let  $f_\omega$  be given by  $f_\omega(x) = f(x + \omega)$ . Then,  $f_\omega$  is supported in the translated rectangle  $\theta' - \omega \ni 0$ . Note that the dual of a translated rectangle is the same as that of the original rectangle.  $\widehat{f}_\omega$  is given by  $\widehat{f}_\omega(\xi) = e^{2\pi i \omega \cdot \xi} \widehat{f}(\xi)$  and, by our previous argument, is constant on tile in  $\Pi_{\theta'}$ . Hence,  $|\widehat{f}| = |\widehat{f}_\omega|$  is also constant for any  $T \in \Pi_{\theta'}$ .

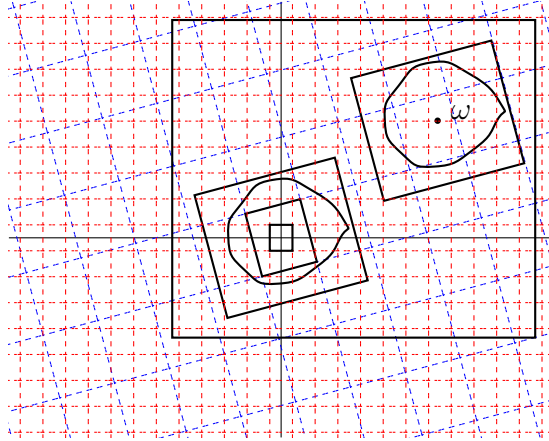


FIGURE 4. The support of  $f$  containing  $\omega$ , the support of  $f_\omega$ ,  $\theta$ ,  $\theta'$  and  $\theta' - \omega$  along with the duals and tilings:  $\theta^*$ ,  $\theta'^*$ ,  $\Pi_\theta$  (dashed red) and  $\Pi_{\theta'}$  (dashed blue).

The smaller the rectangle enclosing the support of  $\widehat{f}$ , the larger the areas on which  $\widehat{f}$  and its modulus are constant and vice versa i.e. heuristically, more concentrated functions are transformed to more spread out functions and vice versa. This concludes our heuristic investigation of the Fourier transformation.

### SOBOLEV SPACES

We would like to measure, in some sense, the size of a function; one way to do this is with the Sobolev norm. Recall that for an  $n$ -tuple  $\alpha = (\alpha_1, \dots, \alpha_n)$ , we have that  $\partial^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$  and  $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$ . For non-negative integer  $k$  and  $p \in [1, \infty]$ , the Sobolev Space  $W^{k,p}(\mathbb{R}^n)$  is the completion of  $\mathcal{C}_c^k(\mathbb{R}^n)$  with the Sobolev norm

$$\|f\|_{W^{k,p}(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} \sum_{|\alpha| \leq k} |\partial^\alpha f|^p dx \right)^{\frac{1}{p}} \tag{8}$$

**Sobolev Embedding Theorem.** If  $kp > n$ , then  $W^{k,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$  as

$$\|f\|_{L^\infty(\mathbb{R}^n)} \lesssim \|f\|_{W^{k,p}(\mathbb{R}^n)} \tag{9}$$

**Example.** Suppose  $n = k = p = 1$ . Choose  $f \in W^{1,1}(\mathbb{R})$ . Then, by the fundamental theorem of calculus,

$$|f(x)| = \left| \int_{-\infty}^x f'(t) dt \right| \leq \int_{-\infty}^x |f'(t)| dt \leq \|f'\|_{L^1(\mathbb{R})} \quad (10)$$

And thus,

$$\|f\|_{L^\infty(\mathbb{R})} = \sup_{x \in \mathbb{R}} |f(x)| \leq \|f'\|_{L^1(\mathbb{R})} \quad (11)$$

As indicated by the Sobolev Embedding Theorem, in higher dimensions the Sobolev inequality might not be true in the borderline case  $kp = n$ ; its failure is related to some interesting examples in conformal geometry.

**Example.** Suppose  $p = 2$  and  $k > \frac{n}{2}$ . Then

$$\|f\|_{W^{k,2}(\mathbb{R}^n)} \approx \left( \int_{\mathbb{R}^n} (1 + |\xi|^2)^k |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \quad (12)$$

And hence, it follows via the Fourier inversion and Cauchy-Schwarz inequality that

$$\|f\|_{L^\infty(\mathbb{R}^n)} \lesssim \|f\|_{W^{k,2}(\mathbb{R}^n)} \quad (13)$$

## FRACTIONAL ORDER SOBOLEV SPACES

Recall  $\partial^\alpha f = \int_{\mathbb{R}^n} (2\pi i \xi)^\alpha \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$ . We will now make an educated guess as to how to extend our definition of Sobolev spaces. For  $\alpha \in [0, \infty) \setminus \mathbb{N}$ , let

$$|D|^\alpha f(x) := \int_{\mathbb{R}^n} |2\pi i \xi|^\alpha \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi \quad (14)$$

And thus, we define

$$W^{\alpha,p}(\mathbb{R}^n) := \{f \in L^p(\mathbb{R}^n) : |D|^\alpha f \in L^p(\mathbb{R}^n)\} \quad (15)$$

We will now use a Littlewood-Paley decomposition to help us measure the size of a function. Let us construct a partition of unity adapted to the open cover of annuli and a ball  $\{B_{2^{n+1}}(0) \setminus \overline{B_{2^{n-1}}(0)} : n \in \mathbb{N}_0\} \cup \{B_1(0)\}$ . Choose some  $\eta \in C_c^\infty(\mathbb{R}^n)$  such that  $\eta \equiv 1$  on  $B_{\frac{1}{2}}(0)$ ,  $\eta \equiv 0$  outside  $B_1(0)$ . Then,  $\{\eta_n = \eta(2^{-n}\xi) - \eta(2^{1-n}\xi) : n \in \mathbb{N}\} \cup \{\eta\}$  is such a partition of unity as for all  $\xi \in \mathbb{R}^n$

$$\eta(\xi) + \sum_{n=1}^{\infty} [\eta(2^{-n}\xi) - \eta(2^{1-n}\xi)] = \lim_{n \rightarrow \infty} \eta(2^{-n}\xi) = 1 \tag{16}$$

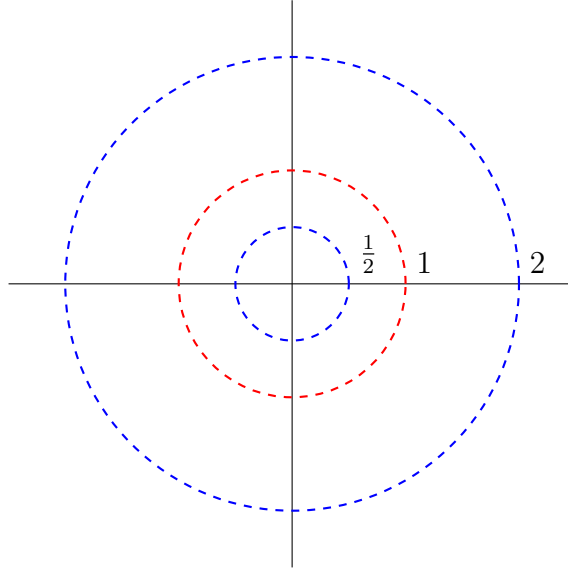


FIGURE 5. An annulus  $B_2(0) \setminus \overline{B_1(0)}$  (blue) and ball  $B_1(0)$  (red) in our open cover of  $\mathbb{R}^2$ .

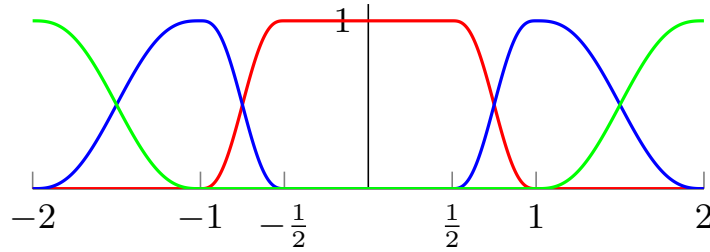


FIGURE 6.  $\eta$ ,  $\eta_1$  and  $\eta_2$  on  $\mathbb{R}$ .

For  $n \in \mathbb{N}$ , we define the decompositions

$$Pf := (\widehat{f} \cdot \eta)^\vee \quad P_n f := (\widehat{f} \cdot \eta_n)^\vee \tag{17}$$

Where the  $\vee$  notation denotes the Fourier inversion. For all  $k \in \mathbb{N}_0$  with  $l \notin \{k - 1, k, k + 1\}$ , we have that  $P_l P_k f = 0$ . Heuristically, we can say of the decompositions that firstly,  $D^\alpha P_\alpha f \sim 2^{k\alpha} P_k f$  and secondly, that  $\widehat{f} = \sum \widehat{P_k f}$  implies that  $f = \sum P_k f$  which in turn implies  $\|f\|_{L^p(\mathbb{R}^n)} \leq \|\sum |P_k f|\|_{L^p(\mathbb{R}^n)}$ . We say this is an  $\ell^1$ -norm type result as the  $\sum |P_k f(x)| = \|\{P_k f(x)\}_k\|_{\ell^1(\mathbb{N})}$  for each  $x \in \mathbb{R}^n$ . A stronger  $\ell^2$ -norm type result can be obtained via Paley-Littlewood theory:

**Theorem.** For  $\alpha \geq 0$  and  $p \in (0, \infty)$  we have that

$$\|f\|_{W^{\alpha,p}(\mathbb{R}^n)} \approx \left\| \left( \sum_{k \geq 0} |2^{k\alpha} P_k f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \quad (18)$$

**Fact.** For  $\alpha > 0$ ,  $p \in (0, \infty)$ , we have that  $\|P_k f\|_{W^{\alpha,p}(\mathbb{R}^n)} \approx 2^{k\alpha} \|P_k f\|_{L^p(\mathbb{R}^n)}$ . This is equivalent to: for  $\alpha \geq 1$  and  $1 < p < \infty$ , if  $\hat{f}$  is supported in an annulus with  $|\xi| \approx R$  and  $R > 1$ , then  $\|f\|_{W^{\alpha,p}(\mathbb{R}^n)} \approx R^\alpha \|f\|_{L^p(\mathbb{R}^n)}$ .

### THE WAVE EQUATION

Solutions to the wave equation in  $\mathbb{R}_x^n \times \mathbb{R}_t$  satisfy:

$$\partial_t^2 u = \Delta_x u \quad u|_{t=0} = f \quad \partial_t u|_{t=0} = 0 \quad (19)$$

And have the form:

$$u(x, t) = \frac{1}{2}(u_+ + u_-) \quad u_\pm(x, t) = \int \hat{f}(\xi) e^{2\pi i(\pm t|\xi| + x \cdot \xi)} d\xi \quad (20)$$

In the last lecture, we spoke about the Bochner-Riesz conjecture. It is in fact a consequence of a harder conjecture related to solutions to the wave equation:

**Local Smoothing Conjecture (Sogge, 1991).** Suppose that  $u$  is a solution to the wave equation and  $\hat{f}$  is supported in an annulus of size  $R$ . Then, for any  $\epsilon > 0$

$$\|u\|_{L^p(\mathbb{R}^n \times [1,2])} \lesssim_\epsilon R^\epsilon \|f\|_{L^p(\mathbb{R}^n)} \quad (21)$$

The conjecture was proved in two dimensions by Guth, Wang and Zhang (2020). It is unverified in higher dimensions. We will cover some fixed time bounds for  $u(\cdot, t)$  by Peral and Miyachi, which were later extended by Stein, Sogge and Seeger.