HARMONIC ANALYSIS: LECTURE FIVE

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Maximal Spherical Averages

In the previous lecture, we began an investigation into the implications of the local smoothing conjecture for the wave equation (which are mapped out on page 9). We found that the the Bochner-Riesz conjecture is related to and is, in fact, a consequence of the local smoothing conjecture. We found this using a variant of the Bochner-Riesz operator which an equivalent formulation (derived using complex analytical methods) involving the wave operator.

$$S^{\alpha} f(x) = \int_{\mathbb{R}^n} (1 - |\xi|^2)_+^{\alpha} \widehat{f}(\xi) e^{2\pi i x \cdot \xi}\, d\xi \tag{1}$$

$$\underline{S}^{\alpha} f(x) = \int_{\mathbb{R}^n} (1 - |\xi|)_+^{\alpha} \widehat{f}(\xi) e^{2\pi i x \cdot \xi}\, d\xi = \frac{1}{C_\alpha} \int_{\mathbb{R}} \frac{e^{-it}\left(e^{it\sqrt{-\Delta}} f(x)\right) - f(x)}{t^{1+\alpha}} e^{ix\cdot\xi} dt \tag{2}$$

$$\|S^{\alpha} f\|_{L^{\frac{2n}{n-1}}} \lesssim \|\underline{S}^{\alpha} f\|_{L^{\frac{2n}{n-1}}} + \|f\|_{L^{\frac{2n}{n-1}}} \lessapprox \|f\|_{L^{\frac{2n}{n-1}}} \tag{3}$$

We subsequently began an investigation of how even partial confirmation of the local smoothing conjecture can be used to prove the Lebesgue's differentiation theorem via spherical averages. This motivated an inquiry into Stein's spherical maximum theorem (1976), which gives an $L^p(\mathbb{R}^n)$ estimate for the spherical maximum operator in three and higher dimensions, and Bourgain's extension of Stein's theorem to two dimensions (1985), which required a much harder geometric proof.

Spherical Maximum Theorem (Bourgain, 1985). If the dimension $n \geq 2$, then $p > \frac{n}{n-1}$ if and only if

$$\left\| \sup_{t>0} |f * d\sigma_t| \right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \tag{4}$$

We began to prove the weakened spherical maximum theorem (below), which is implied by the spherical maximum theorem. Today, we will continue our efforts by utilising the simplified proof of Bourgain's extension by Mockenhaupt, Seeger and Sogge (1992).

Weakened Spherical Maximum Theorem. If the dimension $n \geq 2$, then $p > \frac{n}{n-1}$ if and only if

$$\left\| \sup_{t \in [1,2]} |f * d\sigma_t| \right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \tag{5}$$

Let's unpack what the notation in these theorems mean. $x + t\mathbb{S}^{n-1} \subset \mathbb{R}^n$ is the sphere in $\mathbb{R}^n$ centred at $x \in \mathbb{R}^n$ with radius t. $d\sigma_t$ is the normalised surface measure on $t\mathbb{S}^{n-1}$ i.e. $\int_{t\mathbb{S}^{n-1}} d\sigma_t = 1$. For almost every $x \in \mathbb{R}^n$, the average of $f : \mathbb{R}^n \to \mathbb{R}$ over $x + t\mathbb{S}^{n-1}$ is given by

$$f * d\sigma_t(x) = \int_{t\mathbb{S}^{n-1}} f(x-y) d\sigma_t(y) \tag{6}$$

However, for a fixed $x \in \mathbb{R}^n$, $\sup_{t\in[1,2]} |f * d\sigma_t(x)|$ is not well defined for all $f \in L^p(\mathbb{R}^n)$. It is however well defined for continuous, compactly supported functions $C_c(\mathbb{R}^n)$ which are dense in $L^p(\mathbb{R}^n)$. So, we extended, using (5), the definition of $\sup_{t\in[1,2]} |f * d\sigma_t(x)|$ from $C_c(\mathbb{R}^n)$ to all of $L^p(\mathbb{R}^n)$ by a density argument. The weakened spherical maximum principle makes this extension meaningful; with it, (when all the theorem's conditions are satisfied) we find that the operator $f \mapsto \sup_{t\in[1,2]} |f * d\sigma_t|$ is bounded on all $L^p(\mathbb{R}^n)$, not just $C_c(\mathbb{R}^n)$.

Proof in Three and Higher Dimensions

We will now continue to prove the weakened spherical maximum theorem. The steps from the last lecture are outlined but the previous lecture should be referred to for details.

Step One. Firstly, we Littlewood-Paley decompose f such that the support of $\widehat{P_0 f}$ is contained in the ball centred on the origin with radius 2 and the support of $\widehat{P_k f}$ is contained in an annulus of radius approximately 2^k when $k \geq 1$.

$$f = P_0 f + \sum_{k=1}^{\infty} P_k f \qquad \text{Supp}\,\widehat{P_0 f} \subset B_2(0) \qquad \text{Supp}\,\widehat{P_k f} \subset \{|\xi| \sim 2^k\} \quad k \geq 1 \tag{7}$$

Armed with our decomposition, we introduce the spherical average operator, which returns the spherical average function.

$$f \mapsto \sum_{k=0}^{\infty} \left\| \sup_{t\in[1,2]} |P_k f * d\sigma_t| \right\|_{L^p(\mathbb{R}^n)} \tag{8}$$

An application of the Minkowski and triangle inequality yields that the spherical average function bounds the quantity of interest in the weakened spherical maximum theorem.

$$\left\| \sup_{t\in[1,2]} |f * d\sigma_t| \right\|_{L^p(\mathbb{R}^n)} = \left\| \sup_{t\in[1,2]} \left| \left(\sum_{k=0}^{\infty} P_k f \right) * d\sigma_t \right| \right\|_{L^p(\mathbb{R}^n)} \leq \sum_{k=0}^{\infty} \left\| \sup_{t\in[1,2]} |P_k f * d\sigma_t| \right\|_{L^p(\mathbb{R}^n)} \tag{9}$$

So, a sufficiently strong bound on each term of the spherical average function will yield the weakened spherical maximum theorem.

Step Two. With the above in mind, let's investigate the possibility of bounding the terms. For $p > 1$, the Hardy-Littlewood maximal operator $M : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, which returns the Hardy-Littlewood maximal function, is given by

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r|} \int_{B_r(x)} |f(y)|\, dy \tag{10}$$

In the last lecture, we found that

$$\sup_{t\in[1,2]} |P_k f * d\sigma_t|(x) \lesssim 2^k Mf(x) \tag{11}$$

And consequently, for $k \geq 0$

$$\left\| \sup_{t\in[1,2]} |P_k f * d\sigma_t| \right\|_{L^p(\mathbb{R}^n)} \lesssim 2^k \|f\|_{L^p(\mathbb{R}^n)} \tag{12}$$

So, it is possible to give a bound, although it is not generally strong enough for our purposes as it blows up as $k \to \infty$. It is suitable bound for $\left\|\sup_{t\in[1,2]} |P_0 f * d\sigma_t|\right\|_{L^p(\mathbb{R}^n)}$, however.

Our goal now is to find sufficiently strong bounds when $k \geq 1$. Bounding an integral (L^p norm) using its supremum is fairly routine but we want to bound a supremum using its integral, which will require some work.

Step Three. Let $d\sigma = d\sigma_1$ and note that $\widehat{d\sigma_t}(\xi) = \widehat{d\sigma}(t\xi)$. We continue our investigation by considering the properties of $\widehat{d\sigma}(\xi)$. The Fourier transform of the spherical surface measure is given by

$$\widehat{d\sigma}(\xi) = \int_{\mathbb{S}^{n-1}} e^{-2\pi i x\cdot\xi} d\sigma(x) = \sum_{\pm} e^{\pm 2\pi i|\xi|} a_{\pm}(\xi) = a_{+}(\xi)e^{2\pi i|\xi|} + a_{-}(\xi)e^{-2\pi i|\xi|} \tag{13}$$

Where $e^{\pm 2\pi i|\xi|} = \cos(2\pi|\xi|) \pm i\sin(2\pi|\xi|)$. Furthermore, as a consequence of $\mathbb{S}^{n-1}$ having $n-1$ non-zero principal curvatures, we have the following bounds with related exponents. For any $\xi \in \mathbb{R}^n$ and multi index α,

$$|a_{\pm}(\xi)| \lesssim (1+|\xi|)^{-\frac{n-1}{2}} \qquad \text{and} \qquad |\partial_\xi^\alpha a_{\pm}(\xi)| \lesssim (1+|\xi|)^{-\frac{n-1}{2}-|\alpha|} \tag{14}$$

For example, in three dimensions $a_{\pm}(\xi) = \pm\frac{C}{|\xi|}$ for some constant C which satisfies the conditions of (14) and, consequently,

$$\widehat{d\sigma}(\xi) = C\frac{e^{2\pi i|\xi|} - e^{-2\pi i|\xi|}}{|\xi|} = C\frac{\sin(2\pi|\xi|)}{|\xi|} \tag{15}$$

Similar expressions can be derived for other dimensions using the stationary phase method. Equations (13) and (14) yield a bound for the fourier transform of the spherical surface measure:

$$|\widehat{d\sigma}(\xi)| \leq |a_{+}(\xi)| + |a_{-}(\xi)| \lesssim (1+|\xi|)^{-\frac{n-1}{2}} \leq |\xi|^{-\frac{n-1}{2}} \tag{16}$$

Which, when $t \in [1,2]$, implies that

$$|\widehat{d\sigma_t}(\xi)| = |\widehat{d\sigma}(t\xi)| \lesssim_t |\xi|^{-\frac{n-1}{2}} \tag{17}$$

Combining equations (13) and (17) leads us to the first non-heuristic and genuine approximation given in this class:

$$\widehat{d\sigma}(\xi) = \sum_{\pm} C_{\pm} e^{\pm 2\pi i|\xi|}|\xi|^{-\frac{n-1}{2}} + O(|\xi|^{-\frac{n+1}{2}}) \approx \sum_{\pm} e^{\pm 2\pi i|\xi|}|\xi|^{-\frac{n-1}{2}} + \text{Error} \tag{18}$$

Which yields the following approximations for $f * d\sigma_t$ and $P_k f * d\sigma_t$.

$$f * d\sigma_t(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi)\widehat{d\sigma}(t|\xi|)e^{2\pi i x\cdot\xi}d\xi \approx \int_{\mathbb{R}^n} \widehat{f}(\xi)\frac{e^{2\pi i t|\xi|}e^{2\pi i x\cdot\xi}}{(t|\xi|)^{\frac{n-1}{2}}}d\xi \tag{19}$$

$$P_k f * d\sigma_t(x) \approx \int_{|\xi|\sim 2^k} \frac{\widehat{f}(\xi)e^{2\pi i(t|\xi|+x\cdot\xi)}}{(t|\xi|)^{\frac{n-1}{2}}}d\xi \tag{20}$$

The t in the denominators of the integrands are not so important. By the approximation (20), we can heuristically think of $P_k f * d\sigma_t$ as being of frequency $|\xi| \sim 2^k$ as in the t variable,

$$\frac{d}{dt}\left(P_k f * d\sigma_t(x)\right) \sim 2^k (P_k f * d\sigma_t)(x) \tag{21}$$

Step Four. Now, we will finally give bounds to each term of the spherical average function which are strong enough to prove the weakened spherical maximum theorem. In the last lecture, we used Sobolev embedding in t to give a heuristic explanation as to why

$$\left\| \sup_{t\in[1,2]} |P_k f * d\sigma_t| \right\|_{L^p(\mathbb{R}^n)} \lesssim 2^{\frac{k}{p}} \left\| P_k f * d\sigma_t(x) \right\|_{L^p(\mathbb{R}^n\times[1,2])} \tag{22}$$

would be a reasonable bound. An application of the Fubini-Tonelli theorem yields that

$$\left\| \sup_{t\in[1,2]} |P_k f * d\sigma_t| \right\|_{L^p(\mathbb{R}^n)} \lesssim 2^{\frac{k}{p}} \left\| P_k f * d\sigma_t(x) \right\|_{L^p(\mathbb{R}^n\times[1,2])} \leq 2^{\frac{k}{p}} \sup_{t\in[1,2]} \left\| P_k f * d\sigma_t(x) \right\|_{L^p(\mathbb{R}^n)} \tag{23}$$

Now, we will apply this bound in the case where $p = 2$.

$$\begin{aligned}
\left\| \sup_{t\in[1,2]} |P_k f * d\sigma_t| \right\|_{L^2(\mathbb{R}^n)} &\lesssim 2^{\frac{k}{2}} \sup_{t\in[1,2]} \|P_k f * d\sigma_t\|_{L^2(\mathbb{R}^n)} && [\text{ by (23) }] \\
&= 2^{\frac{k}{2}} \sup_{t\in[1,2]} \left\| \widehat{P_k f}(\xi)\widehat{d\sigma}(t\xi) \right\|_{L^2(\mathbb{R}^n)} && [\text{ by Plancherel's theorem }] \\
&\leq 2^{\frac{k}{2}} \sup_{t\in[1,2]} \left\| \widehat{P_k f} \right\|_{L^2(\mathbb{R}^n)} \sup_{|\xi|\sim 2^k} |\widehat{d\sigma}(t\xi)| \\
&\leq 2^{\frac{k}{2}} (2^k)^{-\frac{n-1}{2}} \|f\|_{L^2(\mathbb{R}^n)} && [\, * \,] \\
&= 2^{-\frac{k(n-2)}{2}} \|f\|_{L^2(\mathbb{R}^n)}
\end{aligned}$$

$$* : \text{ as } \left\| \widehat{P_k f} \right\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\widehat{P_k f}|^2 d\xi = \int_{|\xi|\sim 2^k} |\hat{f}|^2 d\xi \leq \left\| \hat{f} \right\|^2_{L^2} \tag{24}$$

Hence, we prove a special case of the weakened spherical maximum theorem wherein $n \geq 3$ and $p = 2$ (noting that a proof of the necessity of $p > \frac{n}{n-1}$ was sketched in the last lecture) by inserting the inequality given above into (9):

$$\left\| \sup_{t\in[1,2]} |f * d\sigma_t| \right\|_{L^2(\mathbb{R}^n)} \lesssim \sum_{k=0}^{\infty} 2^{-\frac{k(n-2)}{2}} \|f\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)} \tag{25}$$

We will now extend this result to all $p \in (1, 2)$ via the Riesz–Thorin Interpolation theorem applied twice simultaneously.

Riesz–Thorin Interpolation Theorem. Suppose $T : L^{p_0} \to L^{p_0}$ with norm A_0 and $T : L^{p_1} \to L^{p_1}$ with norm A_1 is linear. Then, $T : L^p \to L^p$ with norm $A_0^{\theta} A_1^{1-\theta}$ if $\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}$ for $\theta \in (0, 1)$.

$f \mapsto \sup_{t\in[1,2]} |P_k f * d\sigma_t|$ is not linear but one can linearize by considering the operator

$$Tf(x) := P_k f * d\sigma_{t(x)}(x)$$

where $t(x)$ is a any measurable function of x.

Set $p_1 = 2$ and choose some p_0 arbitrarily close to 1 i.e. $p_0 = 1^+$. For the operator T, $A_0 \lesssim 2^k$ and $A_1 \lesssim 2^{-\frac{k(n-2)}{2}}$ by (12) and (24), respectively. So, for $p \in (p_0, 2)$ and

$$\theta = \frac{\frac{1}{p} - \frac{1}{p_1}}{\frac{1}{p_0} - \frac{1}{p_1}} = \frac{\frac{1}{p} - \frac{1}{2}}{\frac{1}{p_0} - \frac{1}{2}} \tag{26}$$

we have, by the Riesz–Thorin interpolation theorem, that

$$\begin{aligned}\left\|\sup_{t\in[1,2]}|P_kf*d\sigma_t|\right\|_{L^p(\mathbb{R}^n)} &= \left\|\sup_{t\in[1,2]}\sum_{\pm}(P_kf*d\sigma_t)_\pm\right\|_{L^p(\mathbb{R}^n)}\\ &\le \sum_{\pm}\left\|\sup_{t\in[1,2]}(P_kf*d\sigma_t)_\pm\right\|_{L^p(\mathbb{R}^n)}\\ &\lesssim (2^k)^\theta(2^{-\frac{k(n-2)}{2}})^{1-\theta}\left\|f\right\|_{L^p(\mathbb{R}^n)}\end{aligned}$$

We can extend this result to all $p\in(1,2)$ by a limiting argument where $p_0\to 1$ and $\theta\to\frac{2}{p}-1$ as a consequence. Hence, we have for all $p\in(1,2)$ that $\theta=\theta(p)=\frac{2}{p}-1$ and

$$\left\|\sup_{t\in[1,2]}|f*d\sigma_t|\right\|_{L^p(\mathbb{R}^n)} \lesssim \sum_{k=0}^{\infty}(2^k)^\theta(2^{-\frac{k(n-2)}{2}})^{1-\theta}\left\|f\right\|_{L^p(\mathbb{R}^n)} \tag{27}$$

Note that if $n\le 2$, then we are left with an insufficient bound like that given in (12). Naturally, we must ask when is $\sum_{k=1}^{\infty}(2^k)^{\theta(p)}(2^{-\frac{k(n-2)}{2}})^{1-\theta(p)}<\infty$? This condition is equivalent to $0>\theta(p)-\frac{n-2}{2}(1-\theta(p))\Leftrightarrow 0>2(\frac{1}{p}-\frac{1}{2})-\frac{n-2}{2}(1-2(\frac{1}{p}-\frac{1}{2}))=\frac{n}{p}-1-(n-2)=\frac{n}{p}-(n-1)$. So, this bound holds when $p>\frac{n}{n-1}$ as required by the weakened spherical maximum theorem.

Proof in Two Dimensions

Step Five. As previously noted, Mockenhaupt, Seeger and Sogge give an alternate proof to Bourgain's extension of Stein's spherical maximum theorem to two dimensions which we will use hereafter.

Below, we relate spherical averages to solutions of wave equations. Intuitively, this makes sense as spherical averages come up when solving the wave equation in three dimensions as the Huygen's principle suggests they might.

$$\begin{aligned}P_kf*d\sigma_t(x) &\sim \int_{|\xi|\sim 2^k}\frac{\widehat{f}(\xi)e^{2\pi i(t|\xi|+x\cdot\xi)}}{(t|\xi|)^{\frac{n-1}{2}}}d\xi\\ &\sim 2^{-\frac{k}{2}}\int_{|\xi|\sim 2^k}\widehat{f}(\xi)e^{2\pi i(t|\xi|+x\cdot\xi)}d\xi \quad [\text{ as } t\in[1,2] \text{ so } (t|\xi|)^{\frac{n-1}{2}}\sim 2^{k\frac{n-1}{2}}=2^{\frac{k}{2}}\ ]\\ &= 2^{-\frac{k}{2}}e^{it\sqrt{-\Delta}}P_kf(x) \qquad [\text{ the half-wave propagator }]\end{aligned}$$

The local smoothing conjecture in $\mathbb{R}^n$ supposes that integrating over a local time interval such as $[1, 2]$, we get

$$\left\| e^{it\sqrt{-\Delta}} P_k f \right\|_{L^p(\mathbb{R}^n \times [1,2])} \lesssim (2^k)^{(n-1)(\frac{1}{2}-\frac{1}{p})-\frac{1}{p}} \|f\|_{L^p(\mathbb{R}^n)} \tag{28}$$

Lets compare this with the fixed time estimates that we know to be true. Averaging locally in t creates a greater smoothing effect from the wave equation. If $p > \frac{2n}{n-1}$ and $p > 2$, then

$$\sup_{t\in[1,2]} \left\| e^{it\sqrt{-\Delta}} P_k f \right\|_{L^p(\mathbb{R}^n)} \lesssim (2^k)^{(n-1)(\frac{1}{2}-\frac{1}{p})} \|f\|_{L^p(\mathbb{R}^n)} \tag{29}$$

If $n = 2$ and $p > 4$, local smoothing conjecture yields

$$\left\| e^{it\sqrt{-\Delta}} P_k f \right\|_{L^p(\mathbb{R}^2 \times [1,2])} \lesssim (2^k)^{\frac{1}{2}-\frac{1}{p}-\frac{1}{p}} \|f\|_{L^p(\mathbb{R}^2)} \tag{30}$$

This implies that for all $p > 2$, there is some $\varepsilon(p) > 0$ such that

$$\left\| e^{it\sqrt{-\Delta}} P_k f \right\|_{L^p(\mathbb{R}^2 \times [1,2])} \lesssim (2^k)^{(\frac{1}{2}-\frac{1}{p}-\varepsilon)} \|f\|_{L^p(\mathbb{R}^2)} \tag{31}$$

Mockenhaupt, Seeger and Sogge also proved the above independently of the local smoothing conjecture. An outline of the proof follows. Suppose (31) holds for some $p_1 > 2$ and $\varepsilon(p_1) > 0$. Then,

$$\left\| e^{it\sqrt{-\Delta}} P_k f \right\|_{L^{p_1}(\mathbb{R}^n \times [1,2])} \lesssim (2^k)^{\frac{1}{2}-\frac{1}{p_1}-\varepsilon(p_1)} \|f\|_{L^{p_1}(\mathbb{R}^n)} \tag{32}$$

We also get it for $p = 2$ via Plancherel's theorem.

$$\left\| e^{it\sqrt{-\Delta}} P_k f \right\|_{L^2(\mathbb{R}^n \times [1,2])} \lesssim (2^k)^{\frac{1}{2}-\frac{1}{2}-0} \|f\|_{L^2(\mathbb{R}^n)} \tag{33}$$

And $p = \infty$ using the properties of convolutions.

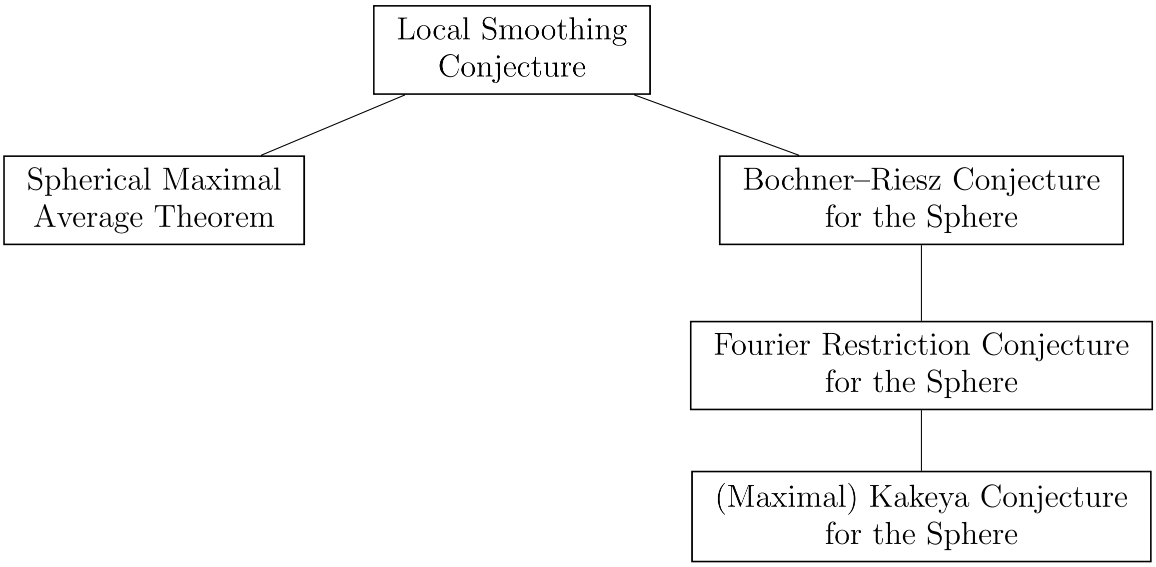
$$\left\| e^{it\sqrt{-\Delta}} P_k f \right\|_{L^\infty(\mathbb{R}^n \times [1,2])} \lesssim \|f\|_{L^\infty(\mathbb{R}^n)} = (2^k)^{\frac{1}{2}-\frac{1}{\infty}-\frac{1}{2}} \|f\|_{L^\infty(\mathbb{R}^n)} \tag{34}$$

Using an interpolation argument, the inequality (31) holds for all $p \in (2, p_1)$. Similarly, the inequality (31) holds for all $p \in (p_1, \infty)$ by interpolating between p_1 and ∞. So our estimate can be extended to all $p > 2$.

$$\begin{aligned}\left\| \sup_{t\in[1,2]} |P_k f * d\sigma_t| \right\|_{L^p(\mathbb{R}^2)} &\le 2^{\frac{k}{p}} 2^{-\frac{k}{2}} \left\| e^{it\sqrt{-\Delta}} P_k f \right\|_{L^p(\mathbb{R}^2\times[1,2])} \\ &\le 2^{\frac{k}{p}} 2^{-\frac{k}{2}} (2^k)^{\frac{1}{2}-\frac{1}{p}-\varepsilon(p)} \|f\|_{L^p(\mathbb{R}^2)} \\ &= 2^{-k\varepsilon(p)} \|f\|_{L^p(\mathbb{R}^2)}\end{aligned}$$

So, as in the proof in dimensions three and higher, the bounds are sufficient to prove the weakened spherical maximum theorem. Thus concludes our investigation of the spherical maximal theorem.

Conjectures in Context



So far in the course, we have seen: the local smoothing conjecture (which has a partial result by Guth, Wang and Zhang) which implies the spherical maximal average theorem and the Bochner-Riesz theorem. The latter implies the restriction conjecture for the sphere which itself implies the Kakeya conjecture for the sphere. We will investigate both of these theorems in the remainder of the course and give their definitions below.

(Dual form of) the Restriction Conjecture for $\mathbb{S}^{n-1}$. This is a completely equivalent formation of the restriction conjecture for $\mathbb{S}^{n-1}$. Define the operator $\mathcal{E}$ for all $f \in L^p(\mathbb{S}^{n-1})$ when $p \ge 1$ by

$$\mathcal{E}f(x) = \int_{S^{n-1}} f(\xi) e^{2\pi i \xi \cdot x} d\xi \tag{35}$$

$\mathcal{E}$ is the adjoint of the operation $f \mapsto \widehat{f}|_{\mathbb{S}^{n-1}}$ i.e. $\langle \mathcal{E}f, g\rangle_{L^2(\mathbb{R}^n)} = \langle f, \widehat{g}|_{\mathbb{S}^{n-1}}\rangle_{L^2(\mathbb{R}^n)}$. Then for all $p > \frac{2n}{n-1}$

$$\|\mathcal{E}f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \tag{36}$$

The (Maximal) Kakeya Conjecture. Suppose $\{T\}_{T\in\mathbf{T}}$ is a family of δ-tubes ($\delta \times \cdots \times \delta \times 1$) in $\mathbb{R}^n$ that point in δ separated directions, then for any non-negative $\{c_T\}_{T\in\mathbf{T}}$ we have

$$\left\| \sum_{T\in\mathbf{T}} c_T \mathbb{1}_T \right\|_{L^{\frac{n}{n-1}}} \lesssim_\varepsilon \delta^{-\varepsilon} \left(\sum_{T\in\mathbf{T}} c_T^{\frac{n}{n-1}} |T| \right)^{\frac{n-1}{n}} \tag{37}$$

where we "pretend" they are not overlapping, they all point in different directions.