Harmonic Analysis Notes

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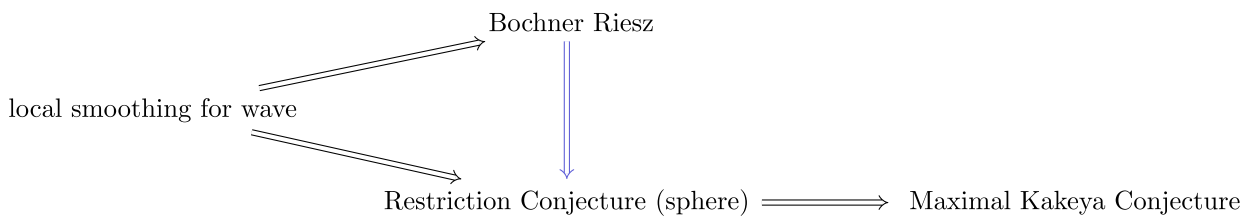
1 Lecture 6

We've previously mentioned the implications:



We will discuss fourier restriction conjecture in more detail today. We will also briefly discuss today some ideas on the implication in blue.

1.1 Motivation

Motivating Question: When does it make sense for us to restrict the fourier transform operator onto a subset, $S \subseteq \mathbb{R}^n$?

If S has lebesgue measure > 0, this is fine, but what if S has measure 0? Recall that in order for us to define the fourier transform for L^p functions, $p \in [1,2]$, we first defined it for $f \in L^1(\mathbb{R}^n)$, before extending it via a limit (since $\mathcal{S}(\mathbb{R}^n) \subseteq L^1 \subseteq L^p$ is dense.)

Since functions in L^p $(p > 1)$ are only defined up to almost-everywhere equivalence, our $\hat{f}$ is not necessarily well defined over a measure 0 set, and we cannot get anything meaningful out of this.

For the case $p = 1$, $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\, dx$ is a bounded, continuous function, and is well defined at every point ξ. Thus, we can make sense of $\hat{f}|_S$, even if S is measure 0 in $\mathbb{R}^n$.

Stein: If $p > 1$, but p is "sufficiently close to 1" then $\hat{f}$ can be restricted to any curved hypersurface in $\mathbb{R}^n$.

We now focus our attention to the case of the sphere. Consider $Rf(\xi) = \hat{f}|_{\mathbb{S}^{n-1}}$, initially defined for $f \in \mathcal{S}(\mathbb{R}^n)$. For $f \in \mathcal{S}(\mathbb{R}^n)$ this is indeed well defined since the fourier transform of a schwartz function is schwartz (and is hence well defined at every point), so the restriction makes sense.

In the same way that we extended fourier transform to L^p functions, if we can establish some uniform bound:

$$\|Rf\|_{L^q(\mathbb{S}^{n-1})} \leq C_{p,q} \|f\|_{L^p(\mathbb{R}^n)}\,,\ \forall f \in \mathcal{S}(\mathbb{R}^n)$$

then, we can clearly extend via continuity the fourier restriction operator, $R : L^p(\mathbb{R}^n) \to L^q(\mathbb{S}^{n-1})$, as a bounded operator.

Thus, we ask for which r, q can we hope for such a bound.

Easiest Case: Consider the case $r = 2$.

Consider $R : L^q \to L^2$. If this is a bounded operator with norm A, then, we can consider the adjoint $R^* : L^2 \to L^{q'}$. (q' here is the dual of q)

We then will have that $R^*R : L^q \to L^{q'}$, with norm A^2. We will call the adjoint $\mathcal{E}$. This is the Fourier extension operator. We compute the definition:

$$\mathcal{E}g(x) = R^*g(x) = (gd\sigma)^\vee(x) = \int_{\mathbb{S}^{n-1}} g(\xi)e^{2\pi ix\cdot\xi}\ d\sigma(\xi)$$

Then, we have that:

$$\begin{aligned} R^*Rf(x) = \mathcal{E}\mathcal{E}^*f(x) = (\hat{f}d\sigma)^\vee(x) &= \int_{\mathbb{S}^{n-1}} \hat{f}(\xi)e^{2\pi ix\cdot\xi}\ d\sigma(\xi) \\ &= f * (d\sigma)^\vee(x) \end{aligned}$$

Thus, we wish to bound convolution with $(d\sigma)^\vee$. Recall from previous lectures, that we have the decay bound of the fourier transform of the sphere:

$$|(d\sigma)^\vee(x)| = |(d\sigma)^\vee(-x)| = |(d\sigma)^\wedge(x)| \lesssim (1+|x|)^{\frac{-(n-1)}{2}}$$

Now, $(1+|x|)^{\frac{-(n-1)}{2}}$ is "almost" in $L^{\frac{2n}{n-1}}$ since the function $(1+|x|)^{-n}$ is "almost" in $L^1(\mathbb{R}^n)$. More formally, it is at least in weak-$L^{\frac{2n}{n-1}}$ space (see the following remark).

Thus, we know that $(d\sigma)^\vee(x)$ is "almost in $L^{\frac{2n}{n-1}}$". We know by Young's convolution inequality, convolution with some (fixed) function in $L^{\frac{2n}{n-1}}$ gives the following bounds:

If g is a function in $L^{\frac{2n}{n-1}}$, then convolution with g is a bounded operator (with norm at most $\|g\|_{L^{\frac{2n}{n-1}}}$.):

- Which sends $L^1 \to L^{\frac{2n}{n-1}}$
- Which sends $L^{\frac{2n}{n+1}} \to L^\infty$

Interpolating between these bounds, we can get all the values connecting these two points (drawn in red in the following diagram). Recall that we wish to obtain a bound on R^*R as an operator from $f \in L^q$ to $L^{q'}$, for some specific q.

To illustrate this, we wish to find q which lies on both the lines:

Remark 1.1. *The above can be formalised as follows.*

There is a weak-type version of Young's convolution inequality (See Theorem 1.4.25. from Classical Fourier Analysis, by Grafakos) , which states as follows.

Theorem 1.2 (Young's Convolution Inequality)**.** *If $1 < p, q, r < \infty$, are such that:*

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$$

Then, there exists constants $C_{p,q} = C_{p,q,r}$ so that, for every $f \in L^p$, and every $g \in L^{q,\infty}$, we have:

$$\|f * g\|_{L^r} \leq C_{p,q}\,\|f\|_{L^p}\,\|g\|_{L^{q,\infty}}$$

We know that:

$$(d\sigma)^\vee(x)^{\frac{2n}{n-1}} \leq (1+|x|)^{-n}$$

so certainly, $(d\sigma)^\vee(x) \in L^{\frac{2n}{n-1},\infty}$. This means that we know that we similarly get bounds (Although we might not get the literal endpoints on the line segment, we can perturb an arbitrarily small amount.) We get at least, that R^*R is a bounded operator from $L^q(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, on the red line minus the two endpoints by interpolation.

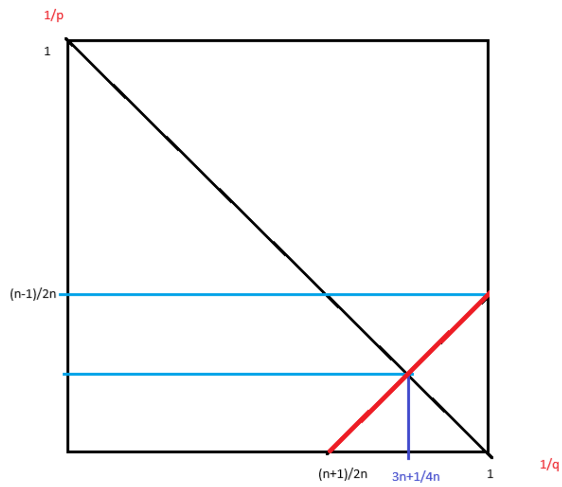


Figure 1: Note the two lines have gradient -1, and 1. Thus their intersection must happen at $\frac{1}{q_n} = \frac{1}{2}\left(\frac{n+1}{2n} + 1\right) = \frac{3n+1}{4n}$. (To see this geometrically, the triangle formed below the lines must be right-isoceles.)

In general, it is slightly easier to work with the fourier extension operator. We will focus entirely on the extension operator formulation from this point onwards.

Definition 1.3. *We will write:*

$$\mathcal{E} : L^p(\mathbb{S}^{n-1}) \to L^q(\mathbb{R}^n)$$

to denote the statement that $\mathcal{E}$ *is a bounded operator between* $L^p(\mathbb{S}^{n-1})$ *to* $L^q(\mathbb{R}^n)$

1.2 Known values

Theorem 1.4 (Tomas-Stein)**.**

$$\mathcal{E} : L^2(\mathbb{S}^{n-1}) \to L^{\frac{2(n+1)}{n-1}}(\mathbb{R}^n)$$

Note that this is often interpolated with the trivial bound:

$$\mathcal{E} : L^1(\mathbb{S}^{n-1}) \to L^{\infty}(\mathbb{R}^n).$$

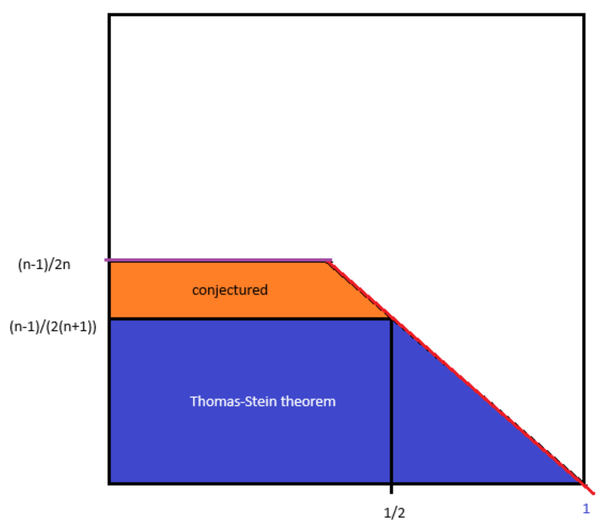
One can also always increase the Lebesgue exponent on $\mathbb{S}^{n-1}$ for any valid extension estimate. For instance:

$$L^{\infty}(\mathbb{S}^{n-1}) \hookrightarrow L^2(\mathbb{S}^{n-1}) \xrightarrow{\quad \mathcal{E} \quad} L^{\frac{2(n+1)}{n-1}}(\mathbb{R}^n)$$

1.3 The Conjecture

Conjecture 1.5 (Restriction Conjecture for Sphere)**.** *$\mathcal{E} : L^q(S^{n-1}) \to L^p(\mathbb{R}^n)$ holds, whenever $p > \frac{2n}{n-1}$ and $q' < \frac{n+1}{n-1}p$*

More generally, there is a version of the restriction conjecture, for any C^2 hypersurface, with strictly positive (Gaussian) curvature, conjectured for the same set of exponents p.



Note that outside of this region, the statement is known to be false. Consider the following two counterexamples.

Counterexample 1: We cannot hope for a result above or equal to the purple line, corresponding to $p < \frac{2n}{n-1}$.

If $p \geq \frac{2n}{n-1}$, then consider $g \equiv 1$, to be identically 1 on the sphere.

We know that $g \in L^q(\mathbb{S}^{n-1})$ (in fact, we can choose the normalisation on the surface measure so that $\|g\|_{L^q(\mathbb{S}^{n-1})} = 1$). Then, we have that:

$$\mathcal{E}g(x) = (d\sigma)^{\vee}(x)$$

but $(d\sigma)^{\vee}(x)$ decays like $(1 + |x|)^{-\frac{n-1}{2}}$. But $\left((1 + |x|)^{-\frac{n-1}{2}}\right)^p$ is integrable only if $p > \frac{2n}{n-1}$.

Counterexample 2 (Knapp): We cannot hope for a result to the right of the red line via the following construction.

Take a small spherical cap, S, of width δ on the unit sphere, and take $g = 1_S$, the indicator function on S.

We know that $\sigma(\delta)$ is on the order of δ^{n-1}. Thus, we have:

$$\|g\|_{L^q(\mathbb{S}^{n-1})} \sim \delta^{\frac{n-1}{q}}$$

However, then, we have:

$$\mathcal{E}g(x) = \int_S e^{2\pi i x \cdot \xi}\, d\sigma(\xi)$$

By local constancy, we can bound $e^{2\pi i x \cdot \xi}$ to be approximately equal to some constant C, for all ξ within the small δ-cap S. Specificially, for x within a radius δ^{-1}, height δ^{-2} tube encapsulating S (let us call this tube T), we have that:

$$\mathrm{Re}(e^{i\theta} e^{2\pi i x \cdot \xi}) \sim C$$

$$|\mathcal{E}g(x)| = |\int_S e^{2\pi i x \cdot \xi}\, d\sigma(\xi)| \gtrsim C\delta^{n-1}$$

Thus, we know that:

$$\|\mathcal{E}g(x)\|_{L^p(\mathbb{R}^n)} \geq \|(\mathcal{E}g(x)) \cdot 1_T\|_{L^p(\mathbb{R}^n)} \geq C\delta^{n-1}\mu(T)^{\frac{1}{p}} = C\delta^{n-1}\left(\delta^{-(n+1)}\right)^{\frac{1}{p}}$$

Thus, in order to achieve a bound on the norm of the restriction operator, we require:

$$\delta^{n-1}\delta^{-\frac{(n+1)}{p}} \lesssim \delta^{\frac{n-1}{q}}$$

regardless of how small δ is. Thus, we require:

$$(n-1) - \frac{n+1}{p} \geq \frac{n-1}{q}$$

and thus:

$$\frac{n+1}{p} \geq \frac{n-1}{q'}$$

(where q' is the dual exponent of q) is necessary.

1.4 Partial results

We know that the restriction conjecture is true for the dimension $n = 2$ case, as proven by Fefferman and Zygmund in the 1970's.

For $n \geq 3$, the conjecture is still currently open in general. The Tomas-Stein theorem, as discussed before, gives us that the conjecture is true for $p > \frac{2(n+1)}{n-1}$, and Stein later proved the endpoint of $p = \frac{2(n+1)}{n-1}$.

Partial results for the dimension 3 case have been shown by Bourgain and Tao, and various others, with the best known current bound is given by Wang and Wu, for $p > \frac{22}{7}$. In the same paper, the restriction conjecture is shown to be true for $p > \frac{154n+6}{77n-95}$, for general n.

1.5 Connection to Bochner-Riesz Conjecture

Theorem 1.6 (Tao)**.** *Suppose* $\forall \alpha > 0$, $\|S^\alpha f\|_{L^p} \lesssim \|f\|_{L^p}$, *whenever* $\frac{2n}{n+1+2\alpha} < p < \frac{2n}{n-1-2\alpha}$.

Then, the fourier extension operator:

$$\mathcal{E}g(x) := \int_{\mathbb{S}^{n-1}} g(\xi)e^{2\pi i x\cdot\xi}\ d\sigma(\xi)$$

satisfies $\forall p > \frac{2n}{n-1}$

$$\|\mathcal{E}g\|_{L^p(\mathbb{R}^n)} \lesssim_p \|g\|_{L^p(\mathbb{S}^{n-1})},$$

Note that this in turn implies the fourier restriction conjecture fully/

Remark 1.7. *To prove the restriction conjecture, it suffices, by a a factorization argument of Maurey, Nikishin and Pisier, to prove the an apparently weaker assertion*

- $\mathcal{E} : L^\infty(\mathbb{S}^{n-1}) \to L^p(\mathbb{R}^n)$ *for all* $p > \frac{2n}{n-1}$

Then it follows that

- $\mathcal{E} : L^p(\mathbb{S}^{n-1}) \to L^p(\mathbb{R}^n)$ *for all* $p > \frac{2n}{n-1}$

Remark 1.8. *The converse of Tao's theorem above is not known for the sphere case (even though many techniques for studying partial results for the restriction conjecture extends and gives us corresponding results for Bochner-Riesz for the sphere). However, it is known that the restriction conjecture for the paraboloid implies Bochner Riesz for the paraboloid (where instead of distances to the sphere, we consider distances to the paraboloid as our smoothing multiplier).*