

# Harmonic Analysis Notes

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## 1 Lecture 9

### 1.1 Motivation

Consider the Schrodinger equation.

$$\begin{cases} i\partial_t u = \Delta_x u \\ u(x, 0) = f(x) \end{cases}$$

If  $f \in L^2(\mathbb{T}^2)$  is given by the fourier series  $\sum_{n \in \mathbb{Z}^2} a_n e^{2\pi i n x}$ , then our  $u$  must be of the form:

$$u(x, t) = \sum_{n \in \mathbb{Z}^2} a_n e^{2\pi i (n x + |n|^2 t)}$$

Just like for the wave equation, we might ask how  $\|u\|_{L^p(\mathbb{T}^2 \times \mathbb{T})}$  compares to  $\|f\|_{L^2(\mathbb{T}^2)}$ .

As usual, the case when  $p = 2$  is trivial. We have:

$$\|u(x, t)\|_{L^2(\mathbb{T}^2 \times \mathbb{T})} = \left\| \sum_{n \in \mathbb{Z}^2, |n| \leq N} a_n e^{2\pi i (n x + |n|^2 t)} \right\|_{L^2(\mathbb{T}^2 \times \mathbb{T})} = \left( \sum_{n \in \mathbb{Z}^2, |n| \leq N} |a_n|^2 \right)^{\frac{1}{2}}$$

Here we conclude the last line using Plancherel's theorem, by observing that  $e^{2\pi i (n x + |n|^2 t)}$  have disjoint fourier support across  $n \in \mathbb{Z}^2$ .

**Question:** Which  $p$  should we look at?

It turns out that the interesting case is when  $p = 4$ , the Tomas-Stein exponent in 3 dimensions.

**Question:** Can we get a bound:

$$\|u\|_{L^4(\mathbb{T}^2 \times \mathbb{T})} \lesssim \|f\|_{L^2(\mathbb{T}^2)}$$

**Answer:** Not for  $\mathbb{T}^2$ .

What if we change to the case where  $f$  has compact fourier support? That is, suppose  $f(x) = \sum_{n \in \mathbb{Z}^2, |n| \leq N} a_n e^{2\pi i n x}$ .

**Question:** How can we maximise  $\|u\|_{L^4(\mathbb{T}^2 \times \mathbb{T})}$ , given that  $\sum_{|n| \leq N} |a_n|^2 = 1$   
There are three ways we can make sense of this problem.

### 1.2 Decoupling

One thing to notice is that the problem is essentially a rescaled version of decoupling for a paraboloid. Instead of  $\delta$ -separated points on the unit disk, we are instead considering 1-separated points on a disk of radius  $N$ .

We can relate this back to decoupling on the paraboloid. Recall the question which decoupling of the paraboloid concerns:

- When we have many small spherical caps (say of radius  $\delta$ ) on the paraboloid, and functions which are fourier supported within these cylindrical boxes which contain these caps. Can we get a bound for the  $L^p$  norm on the sum of these functions, in terms of an  $\ell^2$  sum of the pieces?

We will see that the estimate we want is indeed just a rescaled version. Now, the problem we have at hand is:

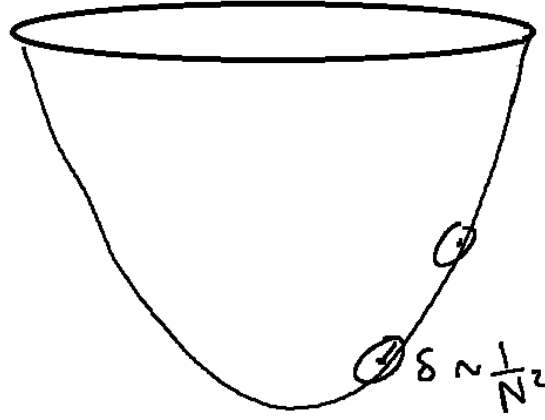
$$\begin{aligned} \|u\|_{L^4(\mathbb{T}^2 \times \mathbb{T})}^4 &= \int \left| \sum_{|n| \leq N} a_n e^{2\pi i(nx + |n|^2 t)} \right|^4 dx dt \\ &= \frac{1}{N^4} \int_{[0, N^2]} \int_{[0, N]^2} \sum_{|n| \leq N} a_n e^{2\pi i(\frac{n}{N}x + (\frac{|n|}{N})^2 t)} \\ &= \frac{1}{N^6} \int_{[0, N^2]} \int_{[0, N^2]^2} \sum_{|n| \leq N} a_n e^{2\pi i(\frac{n}{N}x + (\frac{|n|}{N})^2 t)} \end{aligned}$$

Here, we have the equality to the last line, since the function inside the integral is periodic after every  $N$  in each  $x$ -direction. Thus, by tiling  $N^2$  copies of  $[0, N]^2$  within  $[0, N^2]^2$ , we changed our value of our integral by a factor of  $N^2$ , so dividing by an extra  $N^2$  we restore the original.

Let:

$$u_n(x, t) = 1_{[0, N^2]^3}(x, t) a_n e^{2\pi i(\frac{n}{N}x + \frac{|n|^2}{N^2}t)}$$

Since each of these are supported on  $[0, N^2]^3$ , a cube of side length  $N^3$ , by our heuristic on fourier transforms, we expect the fourier transforms of each of these to be supported within some  $\frac{1}{N^2}$  cube.



Each  $\hat{u}_n$  is mostly concentrated within some radius  $\frac{1}{N^2}$  ball around the point  $(\frac{n}{N}, \frac{|n|^2}{N^2})$  on the paraboloid.

Our original question becomes if we can get the decoupling estimate:

$$\begin{aligned} \|u\|_{L^4(\mathbb{T}^3)} &= \left\| \sum_{|n| \leq N} u_n \right\|_{L^4(\mathbb{R}^3)} \lesssim_\varepsilon N^\varepsilon \left( \sum_{|n| \leq N} \|u_n\|_{L^4(\mathbb{R}^2)}^2 \right)^{\frac{1}{2}} \\ &= N^\varepsilon \left( \sum_{|n| \leq N} |a_n|^2 \right)^{\frac{1}{2}} \end{aligned}$$

### 1.3 Number Theory

$$\begin{aligned}
\|u\|_{L^4(\mathbb{T}^3)}^4 &= \int_{[0,1]^3} \left| \sum_{|n_i| \leq N} a_n e^{2\pi i(nx + |n|^2 t)} \right|^4 dx dt \\
&= \int_{[0,1]^3} \sum_{|n_i| \leq N} a_{n_1} \overline{a_{n_2}} a_{n_3} \overline{a_{n_4}} e^{2\pi i(n_1 - n_2 + n_3 - n_4)x} e^{2\pi i(|n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2)t} dx dt \\
&= \sum_{|n_i| \leq N} a_{n_1} \overline{a_{n_2}} a_{n_3} \overline{a_{n_4}} \int_{[0,1]^3} e^{2\pi i(n_1 - n_2 + n_3 - n_4)x} e^{2\pi i(|n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2)t} dx dt
\end{aligned}$$

After applying Fubini, note that by symmetry, unless  $(n_1 - n_2 + n_3 - n_4) = 0$  and  $(|n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2) = 0$  we have:

$$\begin{aligned}
&\int_{[0,1]^3} e^{2\pi i(n_1 - n_2 + n_3 - n_4)x} e^{2\pi i(|n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2)t} dx dt \\
&= \int_{[0,1]} e^{2\pi i(|n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2)t} dt \int_{[0,1]^2} e^{2\pi i(n_1 - n_2 + n_3 - n_4)x} dx = 0
\end{aligned}$$

since at least either the integral with respect to  $x$  or the integral with respect to  $t$  will be 0.

When  $(n_1 - n_2 + n_3 - n_4) = 0$  and  $(|n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2) = 0$ , we have that piece of the integral is 1 (since the inside function is just constant 1, and we're integrating over  $[0, 1]^3$ ).

Thus, our norm  $\|u\|_{L^4(\mathbb{T}^3)}^4$  corresponds exactly to finding:

$$\sum_{\substack{|n_i| < N, \\ n_1 + n_3 = n_2 + n_4 \\ |n_1|^2 + |n_3|^2 = |n_2|^2 + |n_4|^2}} a_{n_1} \overline{a_{n_2}} a_{n_3} \overline{a_{n_4}}$$

Thus, we can reduce the problem to trying to understand the distribution of solutions to:

$$\begin{aligned}
n_1 + n_3 &= n_2 + n_4 \\
|n_1|^2 + |n_3|^2 &= |n_2|^2 + |n_4|^2
\end{aligned}$$

with  $n_1, n_2, n_3, n_4 \in \mathbb{Z}^2$ , with  $|n_1|, |n_2|, |n_3|, |n_4| \leq N$

### 1.4 Geometric Interpretation

As with the previous section, we can interpret the problem of bounding  $\|u\|_{L^4}^4$  by counting the number of solutions:

$$n_1 + n_3 = n_2 + n_4 \tag{1.4.1}$$

$$|n_1|^2 + |n_3|^2 = |n_2|^2 + |n_4|^2 \tag{1.4.2}$$

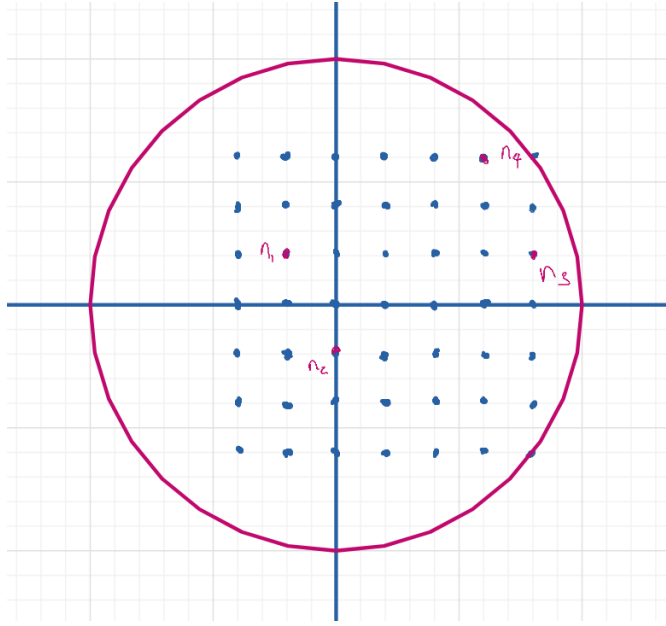
with  $|n_1|, |n_2|, |n_3|, |n_4| \leq N$ .

We can reinterpret the condition above geometrically as follows.

The condition  $|n_1|, |n_2|, |n_3|, |n_4| \leq N$  correspond to  $n_1, n_2, n_3, n_4$  lying within a radius  $N$  circle.

Equation 1.4.1 rearranges to:

$$n_1 - n_2 = n_4 - n_3$$



So the vectors  $n_1 - n_2$  and  $n_4 - n_3$  are identical. In particular, this means that the four points  $n_1, n_2, n_3, n_4$  when plotted in the lattice grid  $\mathbb{Z}^2$ , forms a parallelogram (with vertices in that order), since we know the two opposite sides corresponding to  $n_1 - n_2$  and  $n_4 - n_3$  are parallel and equal in length.

Now, rearranging equation 1.4.2, we know that:

$$\begin{aligned} n_1 \cdot n_1 - n_2 \cdot n_2 &= n_4 \cdot n_4 - n_3 \cdot n_3 \\ (n_1 - n_2) \cdot (n_1 + n_2) &= (n_4 - n_3) \cdot (n_4 + n_3) \end{aligned}$$

But we know  $n_1 - n_2 = n_4 - n_3$ . Thus, we know that:

$$(n_1 - n_2) \cdot (n_1 - n_4 + n_2 - n_3) = 0$$

But  $n_1 - n_4 = n_2 - n_3$  as they are the other two sides of the parallelogram. Thus, we know that  $(n_1 - n_2) \cdot (2(n_3 - n_2)) = 0$  and thus  $(n_1 - n_2)$  is perpendicular to  $n_3 - n_2$ . But this means the four points form a rectangle.

So our conditions on  $n_1, n_2, n_3, n_4$  corresponding to equations 1.4.1 and 1.4.2 correspond exactly to the four points forming a rectangle. Thus, we can reformulate the problem of bounding  $\|u\|_{L^4}^4$ , with bounding the number of rectangles with vertices on integer coordinates that lie within the circle of radius  $N$ .

Using this geometric idea and results from topology (notably, the Szemerédi-Trotter theorem), Sebastien Herr and Beomjong Kwak proved in 2023, the optimal bound:

$$\|u\|_{L^4(\mathbb{T}^3)} \leq C(\log N)^{\frac{1}{4}} \|f\|_{L^2(\mathbb{T}^2)}$$

## 1.5 Lower dimensions

It turns out the question is actually harder to answer for dimension 2. Consider the following analog:

**Question:** For the function given by:

$$u(x, t) = \sum_{n=1}^N a_n e^{2\pi i(n x + n^2 t)}$$

What bound can we get of  $\|u\|_{L^6(\mathbb{T} \times \mathbb{T})}$  in terms of  $\sum_{n=1}^N |a_n|^2$ ?

The conjectured answer is that we can get:

$$\|u\|_{L^6(\mathbb{T} \times \mathbb{T})} \lesssim \log(N)^{\frac{1}{6}} \left( \sum_{n=1}^N |a_n|^2 \right)^{\frac{1}{2}}$$

which is currently still open. However, the following progress has been made:

- Bourgain, Demeter (2015) : We have the bounds:

$$\|u\|_{L^6(\mathbb{T} \times \mathbb{T})} \leq C_\varepsilon N^\varepsilon \left( \sum |a_n|^2 \right)^{\frac{1}{2}}$$

- Guth, Maldague, Wang (2019) : There exists some finite  $c \geq 0$  so that:

$$\|u\|_{L^6(\mathbb{T} \times \mathbb{T})} \lesssim_c \log(N)^c \left( \sum |a_n|^2 \right)^{\frac{1}{2}}$$

- Guo, Zane Li, Yung (2021): For any  $c > 2$ , we can have:

$$\|u\|_{L^6(\mathbb{T} \times \mathbb{T})} \lesssim_c \log(N)^c \left( \sum |a_n|^2 \right)^{\frac{1}{2}}$$

## 1.6 Decoupling of moment curve

We can try generalise it in the following way.

**Question:** For which  $p$  can we get a bound:

$$\left( \int_{[0,1]^d} \left| \sum_{n=1}^N a_n e^{2\pi i(n x_1 + n^2 x_2 + \dots + n^d x_d)} \right|^p dx_1 dx_2 \dots dx_d \right)^{\frac{1}{p}} \lesssim \left( \sum_{n=1}^N |a_n|^2 \right)^{\frac{1}{2}}$$

**Answer:** It turns out we can get a bound whenever  $p = d(d+1)$ .

Note that for  $d = 2$ , this is just the dimension 2 version which we just discussed.

In the same way, we can correspond this bound to counting the number of solutions for the system:

$$\begin{aligned} n_1 + n_2 + \dots + n_{\frac{d(d+1)}{2}} &= n_{\frac{d(d+1)}{2}+1} + \dots + n_{d(d+1)} \\ n_1^2 + n_2^2 + \dots + n_{\frac{d(d+1)}{2}}^2 &= n_{\frac{d(d+1)}{2}+1}^2 + \dots + n_{d(d+1)}^2 \\ &\dots \\ n_1^d + n_2^d + \dots + n_{\frac{d(d+1)}{2}}^d &= n_{\frac{d(d+1)}{2}+1}^d + \dots + n_{d(d+1)}^d \end{aligned}$$

for  $n_1, \dots, n_{d(d+1)} \in [1, N]$ .

History and progress of this problem is as follows:

- $d = 2$  is discussed in the previous section.
- $d = 3$  is solved by Trevor Wooley in 2014 using number theoretic techniques.
- $d \geq 4$  solved by Bourgain, Demeter, Guth in 2015 using Decoupling and multilinear key estimates.
- $d \geq 4$  Alternative proof found by Trevor Wooley in 2018 using number theory
- Shaoming guo, Zane Li, Pavel Zorin, Po-Lam Yung (2023): simplifications to the proof using techniques from Wooley.