Harmonic Report: Szeméredi-Trotter Bounds

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Abstract

In this report, we will discuss the Szemerédi-Trotter theorem. We will first provide a complete elementary proof to, following the proof found by Székly, and as outlined by Tao in [7]. We will discuss how an related result for intersection of tubes (as shown by Demeter and Wang in [2]) might hold. We finish by discussing how the Szemerédi-Trotter theorem can be used to heuristically motivate why a specific case of the Furstenberg set conjecture could hold.

1 Introduction

Given a set of points and a set of lines, one natural question to ask is how many point-line pairs, of which a point lies on one of the lines, there are. For a set of points P, and a set of lines L, we are interested in the number of pairs (p, ℓ) for which $p \in \ell$. Denote the set of all such pairs $I(P, L)$. The Szerémedi-Trotter theorem gives us a bound on the total number of such incidences.

Notation: For the rest of this report, all lines and points will be taken in the plane, $\mathbb{R}^2$. We will use $\#S$ or $|S|$ for a finite set S to denote the number of elements within S. We will denote $|E|_\delta$ to denote the smallest number of δ-balls required to cover E, and use E_δ to denote such a covering. Let $A \lesssim B$ mean that there exists some absolute constant C so that $A \leq CB$. We additionally use $\sim$ to denote two values which are comparable. I will use $\gg$ to informally denote much greater than; usually in the context of one side being able to grow much faster than the other.

2 The (Discrete) Szeméredi Trotter Theorem

We shall give the following elementary and graph theoretic proof of the Szeméredi Trotter Theorem.

Theorem 2.1 (Szeméredi Trotter Theorem)**.** *We have*

$$\#I(P, L) \lesssim \#P^{\frac{2}{3}} \#L^{\frac{2}{3}} + \#P + \#L$$

We will use the following lemma as our main tool (Don't worry, the definitions are right below):

Lemma 2.2 (Crossing Inequality)**.** *For a connected graph, as long as $\#E \geq 4\#V$, we have:*

$$\mathrm{Cr}(G) \geq \frac{(\#E)^3}{64(\#V)^2}$$

Remark 2.3. *There is the better known bound of $\frac{1}{29} \frac{\#E^3}{\#V^2}$, whenever $\#E \gg \#V$, shown by Ackerman in [1].*

Here, we use the following definitions:

Definition 2.4 (Crossing number)**.** For each drawing of a graph, we define the *crossings* as the number of intersections of pairs of edges, outside of the list of vertices.

We define the *crossing number*, $\mathrm{Cr}(G)$ of a graph G as the minimum number of crossings, across all possible drawings of G.

We will assume from this point onwards that each drawing is in generic configuration; that there are no concurrencies of 3 or more lines. We can achieve this by simply perturbing one of the lines slightly. By the way we defined the crossing numbers (as number of intersections between pairs), this does not impact the crossing number.

Without further ado, let us prove the lemma!

Proof of Lemma 2.2. Consider an arbitary connected graph G. If G is planar, then $\mathrm{Cr}(G) = 0$. Otherwise, consider finding a drawing of G with minimal crossings. By removing one of the edges which cross, we note that after at most $\mathrm{Cr}(G)$ edge removals, we can obtain a planar graph G'.

Recall that by Euler's Characteristic formula, we know that for all connected planar graphs, we have (here we count bounded faces only):

$$\#V + \#F - \#E = 1$$

Now, since each face requires at least 3 edges around it, and each edge is adjacent to at most 2 faces, we know that:

$$3\#F \leq 2\#E$$

Thus, re-arranging, we know that:

$$\begin{aligned} \#E - \#F &= \#V - 1 \\ \#E = 3\#E - 2\#E \leq 3\#E - 3\#F &= 3\#V - 3 \end{aligned}$$

and thus we know that at least $\#E \leq 3\#V$.
Thus, by removing the crossed edges as discussed earlier, we know that for any graph G, we have:

$$\begin{aligned} &\#E - \mathrm{Cr}(G) \leq 3\#V \\ &\mathrm{Cr}(G) \geq \#E - 3\#V \end{aligned}$$

Fix a (minimal) drawing of G. Consider now, removing a certain number of vertices. Suppose we uniformly and independently randomly remove vertices, so each vertex has probability p to remain. Call the resulting (random) reduced graph G', with edges E' and vertices V'. Call the resulting drawing (from directly removing the vertices and associated edges of the previous drawing) D'. By earlier, we know that:

$$\mathbb{E}(\mathrm{Crossings}(D')) \geq \mathbb{E}(\mathrm{Cr}(G')) \geq \mathbb{E}(\#E') - 3\mathbb{E}(\#V')$$

since the inequality certainly holds true pointwisely for every possible output of G'.

We know that since each vertex has a probability of p to remain, $\mathbb{E}(\#V') = p(\#V)$. For each edge in the original graph, we know that it will remain in the reduced graph if and only if the two vertices it joins both remains. Each crossing in the drawing in turn remains if and only if both edges involved remains.

Thus, we have:

$$\mathbb{E}(\mathrm{Crossings}(D')) = p^4 \,\mathrm{Cr}(G) \geq \mathbb{E}(\mathrm{Cr}(G')) \geq \mathbb{E}(\#E') - 3\mathbb{E}(\#V') = p^2(\#E) - 3p(\#V)$$

Re-arranging, we have:

$$\mathrm{Cr}(G) \geq p^{-2}(\#E) - 3p^{-3}(\#V)$$

Thus, as long as $\#E \geq 4\#V$, we can pick $p^{-1} = \frac{\#E}{4\#V}$ and obtain:

$$\begin{aligned} \mathrm{Cr}(G) &\geq p^{-2}\left((\#E) - 3p^{-1}(\#V)\right) \\ &= p^{-2}\left((\#E) - \frac{3}{4}(\#E)\right) \\ &= p^{-2}\frac{\#E}{4} = \frac{\#E^3}{64\#V^2} \end{aligned}$$

□

Remark 2.5. *We can also repeat this argument for $\#E \geq (3+\varepsilon)\#V$, to get growth on the order of $\#E^3/\#V^2$ (with a smaller constant). However, we only really care about the case where $\#E$ grows much faster than $\#V$.*

We now prove the Szemerédi-Trotter theorem, as formulated in Theorem 2.1.

Proof. First, consider the subset $L' \subseteq L$ consisting of lines which contain at least two points in P. Since each line that contains at most one point contributes at most 1 incidence, we know that the remaining lines contribute at most $\#L$ incidences. Similarly, we can restrict our attention to points which lie on at least 8 lines, since all other points will contribute at most 7 incidences, so there will be at most $7\#P \sim \#P$ incidences which we removed.

We will show that $\#I(P', L') \lesssim (\#P)^{\frac{2}{3}}(\#L)^{\frac{2}{3}}$. Note that as explained above, this will give us that $I(P, L) \lesssim (\#P)^{\frac{2}{3}}(\#L)^{\frac{2}{3}} + \#P + \#L$.

We treat the configuration of lines and points like a planar graph, with the edges being line segments and points in P being vertices (note that since we only care about incidences, we can chop off the unbounded ends of the lines). For each line in L' with n vertices (note that $n \geq 2$), we will obtain $n - 1 \sim n$-edges in our graph, and n incidences in $I(P', L')$. Thus, by summing over all the lines, we know that $\#E \sim \#I(P', L')$.

Since each pair of lines cross at most once, we know that there are at most $(\#L')^2 \leq \#L^2$ crossings. Now, there are $\#P' \leq \#P$ vertices. By our assumption, each vertex lies on at least 8 lines, so each vertex is of degree at least 8. Thus, since the number of edges is half the sum of the degrees(since each edge joins two vertices) we know that $\#E \geq 4\#V$. Thus by the crossing inequality, we know that:

$$\#L^2 \geq \#\text{ Crossings } \geq \frac{\#E^3}{64\#V^2}$$

Rearranging, we thus have:

$$\#I(P', L')^3 \lesssim \#L^2\#P^2$$
$$\#I(P', L') \lesssim \#L^{\frac{2}{3}}\#P^{\frac{2}{3}}$$

□

2.1 Application to additive combinatorics

One easy application is the following.

Proposition 2.6. *For a finite subset $A \subseteq \mathbb{R}$, we have that:*

$$\max(\#(A + A), \#(A \cdot A)) \gtrsim (\#A)^{\frac{5}{4}}$$

where $A + A = \{a + b \ : \ a, b \in A\}$ and $A \cdot A = \{a \cdot b \ : \ a, b \in A\}$ are sum and product sets respectively.

To see this, consider taking the set of points $P = (A \cdot A) \times (A + A)$, and the set of lines L corresponding to the lines $y = \frac{1}{a}x + b$ where $a, b \in A$. Note that we then have $(\#A)^2$ lines, and $(\#(A + A))(\#(A \cdot A))$ points. Note that trivially, we have $\#P \leq \#A^4$ since $A + A, A \times A$ are each of size at most $\#A^2$.

Then, each line has $\#A$ points from P on it; the line corresponding to $\frac{1}{a}x + b$ (with $a, b \in A$) contains all points of the form $(ac, b + c)$ for $c \in A$. Since c is free to take on any value in A, there are at least $\#A$ such points.

Thus, we have that there are at least $\#A^3$ incidences. However, by the Szemerédi-trotter theorem, we know that:

$$\begin{aligned}(\#A)^3 = I(P, L) &\lesssim \#P^{\frac{2}{3}}\#\text{Ł}^{\frac{2}{3}} + \#P + \#\text{Ł} \\ &= \#P^{\frac{2}{3}}(\#A)^{\frac{4}{3}} + \#P + \#A^2 \lesssim \#P^{\frac{2}{3}}(\#A)^{\frac{4}{3}} + \#A^2\end{aligned}$$

where we used the trivial bound $\#P \leq \#A^4$ from earlier. Continuing on, we thus have:

$$\#P \gtrsim \#\frac{A^3}{(\#A)^{\frac{4}{3}}} = \#A^{\frac{5}{2}}$$

Thus, we know that:

$$\max(\#(A + A), \#(A \cdot A)) \geq (\#(A + A)\#(A \cdot A))^{\frac{1}{2}} = \#P^{\frac{1}{2}} \gtrsim \#A^{\frac{5}{4}}$$

Remark 2.7. *In a similar way to the above, we can show that the bound in Theorem 2.1 is essentially tight; all three terms are necessary.*

By considering many lines which all intersect at the same point, we have that the number of incidences is $\sim \#L$. By considering many points which all lie on one line, we have the number of incidences is $\sim \#P$.

Now, to see that the $\#P^{\frac{2}{3}}\#L^{\frac{2}{3}}$ term is necessary consider taking P to be all the integer points within an $N \times 2N^2$ rectangle, $P = \{1, 2, \cdots N\} \times \{1, 2, \cdots, N^2\}$. There are thus $2N^2$ points.
Let L be all the lines of the form $y = ax + b$ where $a \in \{1, 2, \cdots N\}$ and $b \in \{1, 2, \cdots, N^2\}$. There are N^3 lines. Now, there are at least N incidences on every line, since plugging in any $x = 1, 2, \cdots, N$ gives us a positive integer below $2N^2$.

Thus, there are at least N^4 incidences. So in this case $I(P, L) \gtrsim N^4 \sim (\#P\#L)^{\frac{2}{3}} \gg \#P + \#L \sim N^3$, by considering larger and larger N. Thus, we need a term in bound for which the powers on $\#P, \#L$ sum to at least $\frac{4}{3}$. However, by symmetry, we can dualise points and lines, so any sharp bound which contains $\#P^{\alpha}\#L^{\beta}$ must contain $\#P^{\beta}\#L^{\alpha}$. By AM-GM, the symmetric sum of these unbalanced powers will be larger than $(\#P\#L)^{\frac{2}{3}}$, so this term is certainly necessary.

2.2 Reformulation

We can also obtain the following consequence of Theorem 2.1, involving r-rich points. I want to state this version since this is a variation of this version, which can be stated for small thin tubes (which we will talk about shortly).

Definition 2.8. Given a collection of lines L within $\mathbb{R}^2$, we define the set of *r-rich points* to be the set of points which lie in at least r lines within L. We will denote the collection of r-rich points as $P_{r,L}$ (or just P_r if it is obvious which lines we are talking about).

Corollary 2.9 (Szemerédi-Trotter)**.** *Given a set of lines L and $r \geq 2$, we have:*

$$\#P_{r,L} \lesssim \frac{(\#L)^2}{r^3} + \frac{\#L}{r}$$

Proof. Note that if $\#L \geq rP_{r,L}$, we have that $\#P_{r,L} \leq \frac{\#L}{r}$ and we are done. Thus, without loss of generality, let us assume $\#L \leq rP_{r,L}$.

Observe that each r-rich point must give at least r incidences. Thus, we know by Theorem 2.1 that:

$$r\#P_{r,L} \leq I(\#P_{r,L}, L) \lesssim \#L^{\frac{2}{3}}\#P_{r,L}^{\frac{2}{3}} + \#L + \#P_{r,L}$$

Thus, we know that:

$$\#P_{r,L} \lesssim \frac{1}{r-1}\left(\#L^{\frac{2}{3}}\#P_{r,L}^{\frac{2}{3}} + \#L\right) \sim \frac{1}{r}\left(\#L^{\frac{2}{3}}\#P_{r,L}^{\frac{2}{3}} + \#L\right)$$

Following on, we thus have:

$$\begin{aligned}
(\#P_{r,L})^{\frac{1}{3}} &\lesssim \frac{1}{r}\left(\#L^{\frac{2}{3}} + \frac{\#L}{(\#P_{r,L})^{\frac{2}{3}}}\right) \\
\#P_{r,L} &\lesssim \frac{1}{r^3}\left(\#L^2 + \frac{\#L^3}{(\#P_{r,L})^2}\right) \\
&\leq \frac{\#L^2}{r^3} + \frac{\#L^3}{r\#L^2} = \frac{\#L^2}{r^3} + \frac{\#L}{r}
\end{aligned}$$

Remark 2.10. *Note that this process is actually reversible. Corollary 2.9 is actually equivalent to Theorem 2.1. To see this, observe that Corollary 2.9 is evidently equivalent to the case where all points are r-rich. By splitting up the set of points $\mathcal{P}$ into sets corresponding to how many incidences they give and since the constant in the $\lesssim$ in Corollary 2.9 is absolute and is independent of r, L, by summing over r dyadically, we have that since the "termwise" inequality shown in Corollary 2.9 holds, we obtain Theorem 2.1.*

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2.3 A quick discussion of other fields and higher dimensions

Our proof of the Szemeredi-Trotter theorem relies heavily on the topology of $\mathbb{R}^2$ (we used Euler characteristic, and the crossing number inequality). However, some similar results over other fields exist.

In finite fields with p elements, for a prime p, we can obtain the following result:

Theorem 2.11. *Let $F = \mathbb{Z}/q\mathbb{Z}$ for some prime q. Consider the (two-dimensional) projective plane $P(F^3)$. For a collection of lines L and points P within $P(F^3)$, with $\#P, \#L \lesssim N := |F|^\alpha$ for some $0 < \alpha < 2$, we have:*

$$I(P, L) \lesssim N^{\frac{3}{2}-\varepsilon_\alpha}$$

for some $\varepsilon_\alpha > 0$.

which is slightly better than the trivial bound of $N^{\frac{3}{2}}$ that we can obtain from double counting. This result was proven by Bourgain, Katz and Tao, by first obtaining a result for the sum-product problem in $\mathbb{Z}/p\mathbb{Z}$.

Over $\mathbb{C}$, replacing the real lines with complex lines in $\mathbb{C}^2$, we can obtain the same bound, first shown by Tóth in [8]:

Theorem 2.12 (Tóth, 2003)**.** *Given a finite collection, P of points in $\mathbb{C}^2$, and a finite collection of complex lines L in $\mathbb{C}^2$, let $I(P, L)$ denote incidences; the set of pairs $(p, \ell) \in P \times L$ where $p \in \ell$. Then:*

$$\#I(P, L) \lesssim \#P^{\frac{2}{3}}\#L^{\frac{2}{3}} + \#P + \#L$$

Using this, earlier this year, Jiahe Shen in [5] proved this same bound holds over other characteristic 0 fields as well. The proof of the result is very short, and follows from a construction of incidence preserving maps to $\mathbb{C}$. For an algebraically closed field K, and a set of points P, and a set of lines L, there exists injective maps from which takes these points in $P \subseteq K^2$ to points and L to lines in $\mathbb{C}^2$, which preserves all incidences.

The construction is very quick, so we will go through it. Recall the following result:

Lemma 2.13. *Any characteristic 0 field, which is finitely generated over $\mathbb{Q}$, is isomorphic to a subfield of $\mathbb{C}$*

Now, for a list of points, $(x_i, y_i) \in P$, and a collection of lines $a_jx + b_jy = c_j$, which we will treat as just the tuple (a_j, b_j, c_j). All of these points and lines are characterised by the finite list of data, $x_i, y_i, a_j, b_j, c_j \in K$.

An incidence can be determined exactly from this information; each incidence corresponds to some $(x_i, y_i) \in P$ and $(a_j, b_j, c_j) \in L$ for which:

$$a_jx_i + b_jy_i = c_j$$

Every part of this equation is preserved under a field isomorphism. Thus, choosing a field isomorphism, from the subfield generated by this finite list of data, to some subfield of $\mathbb{C}$ we preserve all incidences. Then, simply applying the previous result by Tóth, we obtain the same bound.

For higher dimensions, clearly if we stick to lines, we still get a bound on the order of $O(\#P^{\frac{2}{3}}\#L^{\frac{2}{3}} + \#P + \#L)$ by simply projecting down to $\mathbb{R}^2$, and if we do not have further restriction, this must also be tight, since we can just use our $\mathbb{R}^2$ examples. There are various further bounds, when the lines are restricted to not concentrate in any particular plane. Notably, Guth in [3] has shown that when L is a set of lines where no more than $\#L^{\frac{1}{2}}$ lines share a plane, we have :

$$I(P, L) \lesssim \#P^{1/2}\#L^{3/4} + \#P^{2/3}\#L^{1/2} + \#L + \#P$$

Another generalisation would be to replace the lines with planes of suitable dimension. In this case, after taking care that we do not count too many points between each pair of intersection, Solymosi and Tao have obtained in 2011 (see [6]) the following result:

Theorem 2.14 (Solymosi, Tao). *Let $d \geq 2k$, and let P and L be finite sets of points and k-planes in $\mathbf{R}^d$. such that any two k-planes in L intersect in at most one point. Then we have $|I(P,L)| \lesssim_{k,\varepsilon} |P|^{2/3+\varepsilon}|L|^{2/3} + |P| + |L|$ for any $\varepsilon > 0$*

Note that this is a subpolynomial amount from the Szemeredi theorem in dimension 2 (which we already discussed, is a tight bound).

3 Generalisation to tubes

As mentioned, there is a variation of the Szemerédi-Trotter theorem that works for tubes. Before we state the result, let us first try to figure out how we might try to make sense of how we might try to get a version for tubes.

We will focus on the formulation as in Corollary 2.9. We restrict ourselves to the unit square $[0,1]^2$, and all sets in space we consider will roughly lie within this square. Suppose we have a collection of $\delta \times 1$ tubes $\mathscr{T}$. Since one might think thin tubes are like lines, we might hope that the tubes cannot intersect each other in too many places, too many times. Clearly, if we simply ask how many points lie in r of the tubes, this does not work; since unlike lines, two properly intersecting tubes will share infinite points. Thus we might want to introduce some notion of discreteness to deal with this.

Consider dividing up the $[0,1]^2$ into a grid with δ-spaced grid lines. We define $P_r(\mathscr{T})$ to be the set of $\delta \times \delta$ grid squares which intersect at least r of the tubes. Intuitively the idea is that for tubes in sufficiently different directions, their intersection will cover too many grid squares. It turns out that we can bound the size of P_r. Note that we do need such a notion of separability, since unlike lines, very close (both directionally and positionally) tubes which intersect nearly everywhere.

Hong Wang and Ciprian Demeter proved in 2024 (see [2]), the following analogous version for tubes:

Theorem 3.1. *Let $s \in (0, 1/2]$. Suppose $\Lambda \subset \mathbb{S}^1$ is a (δ, s)-set consisting of δ-intervals (arcs) with cardinality $\#\Lambda \sim \delta^{-s}$. Suppose that for each δ-arc $\theta \in \Lambda$ there is a $(\delta, 1-s)$-set $\mathscr{T}_\theta$ of δ-tubes in the direction (normal to) θ, with $\#\mathscr{T}_\theta \sim \delta^{-1+s}$. Let $\mathscr{T} = \cup_{\theta\in\Lambda}\mathscr{T}_\theta$. Then for each $v > 0$ there is $C_v > 0$ such that for each $1 \leqslant r \lesssim \delta^{-s}$ we have*

$$\#P_r(\mathscr{T}) \leqslant C_v \delta^{-v} \frac{\#\mathscr{T}^2}{r^3}.$$

Given the page constraints of this report, we cannot hope to outline the entire proof here. However I will try highlight some ideas and some main results, focusing on intuition. First, let us do the main relevant definitions. There will be some other definitions which I will leave out; please refer to the paper in those cases.

First, let us define the relevant separation condition here:

Definition 3.2 ((δ, s, C) set)**.** We say a set $E \subseteq \mathbb{R}^n$ with diameter $\lesssim 1$ is a (δ, s, C)-*set* for $s \in (0, n]$ if

$$|E \cap B(x,r)|_\delta \leq Cr^s|E|_\delta$$

holds, for all balls $B(x,r)$ of radius $1 \geq r \geq \delta$. If C is an absolute constant not chosen based off δ or s, we will simply say E is a (δ, s)-set.

This is really a uniformity condition; by re-arranging, we have that $\frac{|E\cap B(x,r)|_\delta}{|E|_\delta} \leq Cr^s$, so we cannot have too much of E concentrated within a small ball. Certainly, we can also
Essentially, it is saying E concentrates within itself approximately like a dimension $\geq s$ subset of some constant radius ball containing E. Note that if we were to allow $s = 0$, the condition becomes completely trivial, as then simply allowing $C = 1$ works. Note that the right hand side becomes smaller as s gets larger. The strongest condition, $s = n$ corresponds to saying E is approximately a ball of some radius comparable to 1.

Note that we have some corollaries (as indeed mentioned by Demeter and Wang) as follows.

Corollary 3.3. *Suppose we have a fixed $\nu > 0$. Let us have $\Lambda, \mathscr{T}, \mathscr{T}_\theta$ as earlier. Suppose we choose some subset P of the δ grid squares, so that each tube in $\mathscr{T}$ intersects $\sim M$ such squares.*
Then, we must have chosen at least:

$$\#P \gtrsim \delta^{\frac{-1}{2}+\nu} M^{\frac{3}{2}}$$

As long as we restrict to sufficiently small $\delta < \delta_\nu$.

The proof of this will be essentially the same argument as how we will obtain the formulation within the heuristic argument in section 4. Here the condition on δ being sufficiently small is to remove our dependence on ν for C_ν, in Theorem 3.1.

Now, I will attempt to discuss a little bit about the proof of Theorem 3.1, as outlined in section 8 of Demeter and Wang's paper. First, we will need to define another separation condition.

Definition 3.4 ((δ, s, K)-Katz-Tao set)**.** Let $s \in (0, 1/2]$. For $s \in (0, d]$, a non-empty set $E \subset \mathbb{R}^d$ with diameter $\lesssim 1$ is called a *(δ, s, K)-Katz-Tao set* if:

$$|E \cap B(x, r)|_\delta \leqslant K(r/\delta)^s$$

holds for all balls $B(x, r)$ and all r between $1 \geq r \geq \delta$. If $K \sim 1$, we will just say it is (δ, s) Katz-Tao.

Remark 3.5. *This feels quite similar to our previous definition, and indeed they are related. If we have a (δ, s) set E, with the additional condition that $|E|_\delta \sim \frac{1}{\delta}^s$, the E is (δ, s) Katz-Tao. Indeed, we can simply take $r = \delta$ to obtain:*

$$|E \cap B(x, r)|_\delta \leq C \left(\frac{1}{\delta}\right)^s |E|_\delta \sim C$$

Note that consequently, we know that in Theorem 3.1, Λ is in fact (δ, s)-Katz-Tao, and for each direction θ, $\mathscr{T}_\theta$ is $(\delta, 1-s)$-Katz-Tao.

To give the same sort of intuitive "motto" on this condition as well, we can say that a set is Katz-Tao, if, without looking too finely (on scales below δ), is distributed like an s-dimensional subset, within $[0, 1]^n$ (or some comparable ball).

Fix some value of $s \in (0, 1/2]$. Let us define $ST(\delta, K_1, K_2)$ be the smallest constant (depending on δ, K_1, K_2) so that $|P_r(T)| \leq ST(\delta, K_1, K_2)\frac{\delta^{-2}}{r^3}$, over all sets T of tubes which are (δ, s, K_1) katz-tao in direction, and $(\delta, 1-s, K_2)$ Katz-Tao in each direction. If K_1, K_2 are absolute constants comparable to 1, we simply write $ST(\delta)$.
Note that this is well defined and we don't have to worry about r-dependence, since r is bounded by some function of δ; there cannot be too many tubes total; there are at most $\sim \delta^{-s}$ directions, and $\sim \delta^{-(1-s)}$ tubes in each direction.

There are some observations we can make of this new value ST.

- For any initial r_0, consider the δ-squares involved in P_{r_0}. For each square, $p \in P_{r_0}$, count the number of tubes within $\mathscr{T}$ which intersect it. We know the number of tubes which intersect p is at least r_0, and below δ^{-1}. Thus, at least $\frac{1}{\log(\delta^{-1})}$ of the total number of squares have the same number (up to, say, a factor of 2) , r, of tubes intersecting it. Let $P \subseteq P_{r_0}$ be the set of δ-squares with $\sim r$ tubes intersecting it. Up to a $\log(\delta^{-1})$ loss, we can thus reformulate ST in terms of the size of these P.
- Observe that Theorem 3.1 corresponds exactly to the statement when K_1, K_2 are absolute constants. Thus, to prove Theorem 3.1, we wish to show the result $ST(\delta) \lesssim_\varepsilon \delta^{-\varepsilon}$.
- Note that $ST(\delta) \lesssim \delta^{-3}$ through a trivial bound as follows:
  We know $r \lesssim \delta^{-1}$, and there are in total, $\sim \delta^{-2}$ many δ-squares within $[0, 1]$. So we know:

$$|P_r| \lesssim \delta^{-2} \lesssim \delta^{-3}\frac{\delta^{-2}}{r^3}$$

- Note that $ST(\rho) \lesssim ST(\delta)$ if $\rho > \delta$. To see this intuitively, we can essentially "bunch up" δ-tubes to get ρ-tubes. Since ST is a supremum over all configurations, $ST(\delta)$, consider any configuration of ρ tubes. Consider distributing $(\rho/\delta)^s$ δ-tubes uniformly (in direction) within each ρ-tube. Distribute $(\rho/\delta)^{1-s}$ δ-tubes spatially within each ρ-tube as well. So there are ρ/δ number of δ-tubes within each ρ-tube. Now, simply looking at the intersections between the ρ-tubes, we expect the δ-tubes within these ρ-tubes involved to also intersect, so there are at least $\left(\frac{\rho}{\delta}\right)^2$ times as many intersections. A slightly more formalised version of this argument is on page 34 of [2].

Following on from the last point, we might hope we can get some sort of reverse bound (possibly with some loss) between scales as well, and then it will be enough. It turns out that we can obtain some sort of bound. The main lemma used in the proof of Theorem 3.1 is then:

Lemma 3.6 (Main bound). *We have, for all sufficiently small $\varepsilon > 0$:*

$$ST(\delta) \leq C_\varepsilon \max_{\tilde{\delta} \gtrsim \delta^{1-d_\varepsilon}} \left(\frac{\tilde{\delta}}{\delta}\right)^{e_\varepsilon} ST(\tilde{\delta})$$

where $d_\varepsilon, e_\varepsilon$ are values depending only on ε, which both tend towards 0, as $\varepsilon \to 0$.

For the purposes of this report, we will skip over discussing this proof; there are too many technical details to be able to give a short overview. This result corresponds to equation 8.6 in [2], and allows us to do an induction on scales type argument. Instead of proving this result, I will highlight the we can obtain the result, $ST(\delta) \leq C_\nu \delta^{-\nu}$ that we wanted.

Let ν_0 be the infimum value of ν, for which we can get a constant C_ν so that $ST(\delta) \leq C_\nu \delta^{-\nu}$. Such a constant certainly exists; we have already shown that $\nu_0 \leq 3$.
Note that since $\delta^{-\nu}$ is decreasing as ν decreases, we know that for every value of $\nu > \nu_0$, we will then be able to find such a constant C_ν satisfying the inequality. Thus, we simply wish to show that $\nu_0 = 0$. We will do this by constructing a smaller ν' satisfying the above, if $\nu_0 \neq 0$.

Proof. Suppose $\nu_0 \neq 0$. Pick and fix ε so that $e_\varepsilon < \nu_0/2$. Applying Lemma 3.6 to $\tilde{\delta}$, we see that for $\nu > \nu_0$, we have

$$ST(\tilde{\delta}) \lesssim C_\nu \tilde{\delta}^{-\nu}.$$

Applying Lemma 3.6 again, we know that:

$$\begin{aligned} ST(\delta) &\leq C_\varepsilon C_\nu \max_{\tilde{\delta} \gtrsim \delta^{1-d_\varepsilon}} \left(\frac{\tilde{\delta}}{\delta}\right)^{e_\varepsilon} \tilde{\delta}^{-\nu} \\ &\lesssim C_\varepsilon C_\nu \left(\delta^{-d_\varepsilon}\right)^{e_\varepsilon - \nu} \delta^{-\nu}. \end{aligned}$$

Note that $\frac{\nu_0 - e_\varepsilon d_\varepsilon}{1-d_\varepsilon} \geq \nu_0$ since $e_\varepsilon d_\varepsilon < \nu_0/2 d_\varepsilon < \nu_0 d_\varepsilon$. This means that by choosing ν sufficiently close to ν_0, we can get:

$$\begin{aligned} \nu &< \frac{\nu_0 - e_\varepsilon d_\varepsilon}{1 - d_\varepsilon} \\ \nu - (e_\varepsilon - \nu)(-d_\varepsilon) &< \nu_0, \end{aligned}$$

noting that $e_\varepsilon, d_\varepsilon$ are fixed small constants.
Choose and fix such an ν. Then, $C_\varepsilon C_\nu$ is a constant which works for this smaller value, contradicting our minimality. □

Thus, we know that from the lemma, we have that for all $\nu > 0$, we can find a constant so that $ST(\delta) \leq C_\nu \delta^{-\nu}$, and thus, Theorem 3.1 follows.

4 Furstenberg Set Conjecture

The Furstenberg set conjecture for two dimensions is as follows:

Conjecture 4.1 (Furstenberg)**.** *Suppose we have a compact set $E \subseteq \mathbb{R}^2$, and a family of lines L within $\mathbb{R}^2$, such that $\dim_H(L) = t$ (For instance, we can consider the space of all lines as two dimensional, since it can be described by a direction and an intercept point.). Suppose we have also that for each line, $\ell \in L$, we have$\dim_H(E \cap \ell) = s$ (Note that we know that thus $s \leq 1$). Then*

$$\dim_H(E) \geq \min(s+t, \frac{3s+t}{2},\ s+1).$$

Note that this conjecture is solved (with an affirmative answer) by Ren and Wang, as seen in [4]. A specific case as noted by Wolff (see [9]), and one of the original motivations for the conjecture is as follows.

Conjecture 4.2. *Suppose we have a family of lines in $\mathbb{R}^2$, L, which contains a line in every direction. (Note that this means L must have dimension at least 1). Suppose E is a compact set with diameter ~ 1 so that $\dim_H(E \cap \ell) \geq s$, for each line $\ell \in L$. Then, $\dim_H(E) \geq \min(s+1, \frac{3s+1}{2})$.*

So, why would this even be true? Consider the following heuristic where we discretise the problem. For convenience, since we are working heuristically anyway, we will replace the Hausdorff dimension, with the Minkowski dimension.

Suppose that each line is incident to at least M points, where $M \gg 1$ is large. Then, we have at least $M\#L$ incidences. Note that the Szemerédi-Trotter theorem implies that:

$$M\#L \lesssim \left(\#P^2\#L^2\right)^{\frac{1}{3}} + \#P \sim \max(\left(\#P^2\#L^2\right)^{\frac{1}{3}}, \#P).$$

Note that here we can drop the $\#L$ term; the $\#L$ term was only added in the proof of Theorem 2.1 to account for lines which contain at most 1 point.

Thus, at least one of:

$$M\#L \lesssim \left(\#P^2\#L^2\right)^{\frac{1}{3}} \quad \text{or} \quad M\#L \lesssim \#P$$

must be true. Then, simply re-arranging, we obtain that it must be true that at least one of:

$$\#P^2 \geq M^3\#L \text{ or } \#P^2 \geq M^2\#L^2$$

Thus, we obtain that:

$$\#P \geq \min(M^{\frac{3}{2}}\#L^{\frac{1}{2}},\ M\#L)$$

Now, consider the following discretised version: for each scale δ, consider a grid with sidelength δ. Let D_δ be the set of grid squares which intersect E.

We consider a (directionally) δ-separated subset of lines from L, and for each such line ℓ, consider counting the number of δ-grid squares along each line within E. There will be $\sim \delta^{-1}$ lines. Since $E \cap \ell$ is s-dimensional, we expect $E \cap \ell$ to intersect $\sim \delta^{-s}$ grid squares. Since the lines are δ-separated, we cannot have too many shared points between distinct lines.

Thus, putting it together, we certainly have that for each scale δ, we expect at least:

$$\#D_\delta \gtrsim \min(\delta^{\frac{-3s}{2}}\delta^{-\frac{1}{2}}, \delta^{-s}\delta^{-1})$$

Thus, we expect E to be at least $\min(\frac{3s+1}{2},\ s+1) = \min(s+\frac{s+1}{2},\ s+1) = \frac{3s+1}{2}$ dimensional (recalling that $s \leq 1$).

Heuristically, we also expect this bound as per Conjecture **??** to be sharp, by sharpness of Theorem 2.1, as discussed earlier. Indeed, every part of the argument above is reversible.

Remark 4.3. *Compare this to the Kakeya conjecture. The (2-dimensional) version Kakeya conjecture is simply given by the case* $s = 1, t = 1$.

Remark 4.4. *There is a similar generalisation to Conjecture 4.1. If we have a collection of lines (with some restrictions on concentration), so that the dimension of the collection (as considered as a subset of the affine space) is at least* $2t$*, and a set* E*, for which the* E *is at least dimension* s *along each line, what can we say about the dimension of* E*?*

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