Harmonic analysis

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1 Course information

The assessment criteria is

1. 20% participation in class (including presentation)
2. 30% note taking (due before the following Friday)
3. 40% final paper
4. 10% mini oral defense

The class will cover recent progress about:

- Bochner-Riesz summability of Fourier series
- Fourier restriction conjecture
- Kakeya conjecture
- Local smoothing conjecture for the wave equation
- Decoupling estimates

2 Lecture 1

Definition 1. *Let $f \in L^1(\mathbb{R}^n)$. We define the* ***fourier transform*** *of f to be*

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx \quad \forall \xi \in \mathbb{R}^n$$

If $f \in L^1(\mathbb{R}^n)$ and $\hat{f} \in L^1(\mathbb{R}^n)$, then we have the following fourier inversion formula

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi \quad \forall x \in \mathbb{R}^n$$

Note that $e^{2\pi ix\cdot\xi} = cos(2\pi ix\cdot\xi) + i\sin(2\pi ix\cdot\xi)$. If we fix ξ and plot it with respect to x_1 and x_2, the graph resembles plane waves travelling in the ξ direction with speed $|\xi|$. What the fourier inversion tells us is if we have a "nice" function then we can represent it as a superposition of plane waves.

The following is a class of functions which will often useful

Definition 2. *We define the* ***Schwartz functions*** $\mathcal{S}(\mathbb{R}^n)$ *to be the space of functions* f *such that* $f \in C^\infty(\mathbb{R}^n)$ *and*

$$|\partial^\alpha f(x)| \lesssim \frac{C_{\alpha,N}}{(1+|x|)^N}$$

If a function is Schwartz, then we may also call it ***rapidly decreasing***

Example 1. *The gaussian* $e^{-\pi|x|^2}$ *is rapidly decreasing.*

Q: Can we talk about Fourier transforms of functions for $f \in L^p(\mathbb{R}^n)$ for $p > 1$?

A: For $p = 2$. Suppose $f \in L^2(\mathbb{R}^n)$. We can take $f_j \in L^1 \cap L^2(\mathbb{R}^n)$ such that $f_j \to f \in L^2(\mathbb{R}^n)$. Then $\hat{f}_j$ converges in $L^2(\mathbb{R}^n)$. Thus we define $\hat{f} := \lim_{j\to\infty} \hat{f}_j$ in L^2. Then

$$\int_{\mathbb{R}^n} |f(x)|^2 dx = \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 d\xi \quad \forall f \in L^2(\mathbb{R}^n)$$

In particular if we define $S_R f(x) := \int_{|\xi|\leq R} \hat{f}(\xi) e^{2\pi ix\cdot\xi} d\xi$, then $S_R f \to f$ in $L^2(\mathbb{R}^n)$.

Remark 1. *We can then define* $\hat{f}$ *for every* $f \in L^1 + L^2$. *In fact one can show that*

$$||\hat{f}||_{L^\infty(\mathbb{R}^n)} \leq ||f||_{L^1(\mathbb{R}^n)}$$
$$||\hat{f}||_{L^2(\mathbb{R}^n)} = ||f||_{L^2(\mathbb{R}^n)}$$

Theorem

(Riesz-Thorin complex interpolation): Suppose $T : L^{p_1}(\mathbb{R}^n) \to L^{q_1}(\mathbb{R}^n)$ and $T : L^{p_2}(\mathbb{R}^n) \to L^{q_2}(\mathbb{R}^n)$ is an operator that is bounded and linear, then $T : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ whenever $\exists\theta \in (0,1)$ s.t. $\left(\frac{1}{p}, \frac{1}{q}\right) = (1-\theta)\left(\frac{1}{p}_1, \frac{1}{q}_1\right) + \theta\left(\frac{1}{p}_2, \frac{1}{q}_2\right)$

Corollary 1. *Let* p' *be the lesbegue exponent of* p. *Then*

$$||\hat{f}||_{L^{p'}(\mathbb{R}^n)} \leq ||f||_{L^p(\mathbb{R}^n)} \quad \forall f \in L^p(\mathbb{R}^n)$$

Now we can define for $f \in L^p$ for $1 \leq p \leq 2$.

$$S_R f(x) = \int_{|\xi| \leq R} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$$

Q: For $1 \leq p < 2$ we ask

1. Is it true that $\forall f \in L^p$, $S_R f \to f$ as $R \to \infty$ in L^p?

2. Is it true that $\forall f \in \mathcal{S}(\mathbb{R}^n)$, we have $||S_1 f||_{L^p(\mathbb{R}^n)} \lesssim_{p,n} ||f||_{L^p(\mathbb{R}^n)}$?

Prop. 1. *$S_R g \to g$ in L^p if $g \in \mathcal{S}$.*

A: A positive answer to the second question implies a positive answer to the first question. Suppose $||S_1 f||L^p \lesssim ||f||_{L^p}$ for some $p \in [1, 2)$. Then there exists $c \in \mathbb{R}$ s.t. $||S_R f||_{L^p} \leq c||f||_{L^p} \quad \forall f \in \mathcal{S}(\mathbb{R}^n)\ \forall R > 0$. Note that if $f \in L^p$, we can choose $f_j \in \mathcal{S}(\mathbb{R}^n)$ s.t. $f_j \to f$ in $L^p(\mathbb{R}^n)$. Then

$$\begin{aligned} ||S_R f - f||_{L^p} &\leq ||S_R f - S_R f_j||_{L^P} + ||S_R f_j - f_j||_{L^P} + ||f_j - f||_{L^p} \\ &\leq (1 + c)||f_j - f||_{L^p} + ||S_R f_j - f_j||_{L^p} \end{aligned}$$

Conversely if $S_R f \to f$ in L^p for every $f \in L^p$, then by uniform boundedness principle there exists $c > 0$ s.t. $||S_R||_{L^p \to L^p} \leq c$. In particular $||S_1||_{L^p \to L^p} \leq c$.

In fact, the two questions above are morally equivalent, in that if we consider the corresponding question for summability of Fourier series in the n dimensional torus, then they are really equivalent by the uniform boundedness principle (the issue with this equivalence on $\mathbb{R}^n$ is because we need S_R to be bounded on $L^p(\mathbb{R}^n)$ for every R before we can apply the uniform boundedness principle; but as we will see below, S_R is usually unbounded on $L^p(\mathbb{R}^n)$).

We focus on question 2, which also makes sense for $p > 2$.

Case 1 $n = 1$**:** Let $f \in \mathcal{S}(\mathbb{R})$, then $S_1 f(x) = \int_{-1}^{1} \hat{f}(\xi) e^{2\pi i x \xi} d\xi = f * K(x)$ where $K(x) = \int_{-1}^{1} e^{2\pi i x \xi} d\xi = \frac{sin(2\pi x)}{\pi x}$. This is almost in L^1. If K were in L^1 then $||f * K||_{L^P} \leq ||f||_{L^P}||K||_{L^1}$.

By utilising the theory of singular integrals, specifically Hilbert transforms with $K(x) = \frac{sin(2\pi x)}{\pi x}$ we have the following corollary.

Corollary 2. *If $f \in L^p(\mathbb{R})$, $1 < p \leq 2$ then $\int_{-R}^{R} \hat{f}(\xi) e^{2\pi i x \xi} d\xi \to f(x)$ in $L^2(\mathbb{R})$ as $R \to \infty$.*

On the other hand if $p = 1$, then $f * K$ is not bounded in L^1 (one can see this by essentially testing against the delta function at 0). Thus if $f \in L^1(\mathbb{R})$ then it may not be true that $\int_{-R}^{R} \hat{f}(\xi) e^{2\pi i x \xi} d\xi \to f$ in $L^1(\mathbb{R})$.

Case 2 $n \geq 2$**:**

$$S_R f(x) = \int_{|\xi| \leq R} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$$

The difficulty with trying to do this on a ball for $n \geq 2$ comes with the curvature of the boundary. If $f \in \mathcal{S}(\mathbb{R}^n)$ and consider $S_1 f(x)$ then $S_1 f(x) = f * K$ where $K(x) = \int_{|\xi| \leq 1} e^{2\pi i x \cdot \xi} d\xi$ satisfies $|K(x)| \lesssim (1+|x|)^{-\frac{x+1}{2}} \quad \forall x \in \mathbb{R}^n$. In fact,

$$K(x) \sim \frac{e^{\pm i 2\pi |x|}}{|x|^{(n+1)/2}} + \text{ faster decaying error term}$$

Herz (1954) used this asymptotics and showed that if $1 \leq p \leq \frac{2n}{n+1}$ then $||S_1 f||_{L^p} \not\lesssim ||f||_{L^p}$. The counterexample is given by $f(x) = (1-|x|^2)_+^{\alpha}$ where $\alpha < \frac{n+1}{2n}$ (in fact, it would have been simpler to just take $f =$ characteristic function of a unit ball). In particular $S_R f \not\to f$ in $L^p(\mathbb{R}^n)$ if $1 \leq p \leq \frac{2n}{n+1}$.

What about $\frac{2n}{n+1} < p < 2$? Is $S_R f \to f$ in L^p. But Fefferman (1971) proved the answer is no. He used the existence of **Kakeya sets** and probabilistic arguments. This thereom is called the **Ball multiplier theorem**.

To summarise, if $n \geq 2$, $p \neq 2$ then $f(x) \neq$ the L^p limit of $\int_{|\xi| \leq R} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$. This leads us to the **Bochner-Riesz summarbility**. For $\alpha > 0$, $R > 0$ we define

$$S_R^{\alpha} f(x) = \int_{|\xi| \leq R} \left(1 - \frac{|\xi|^2}{R^2}\right)^{\alpha} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$$

Q: If $n \geq 2$, for which p do we have

$$||S_1^{\alpha} f||_{L^p} \lesssim ||f||_{L^p}$$

We call this inequality BR_{α}.

A: If BR_{α} holds for a certain p, then $f = L^p-$limit $S_R^{\alpha} f$ for all $f \in L^p$. It turned out that $S_1^{\alpha} f = f * K_{\alpha}$ where $K_{\alpha}(x) = \int_{|\xi| \leq 1} (1-|\xi|^2)^{\alpha} e^{\pm 2\pi i x \cdot \xi} d\xi$. Here $K_{\alpha}(x) \sim \frac{e^{2\pi i |x|}}{|x|^{(n+1)/2+\alpha}} +$ error. We can modify what Herz did: BR_{α} fails if $1 \leq p \leq \frac{2n}{n+1+2\alpha}$.

Conjecture 1. ***(Bochner Riesz)*** *If* $n \geq 2$, $\alpha > 0$, $\frac{2n}{n+1+2\alpha} < p < \frac{2n}{n-1-2\alpha}$ *then* BR_{α} *holds.*

Equivalently let $\varphi_{\delta}(\xi)$ be a smooth function supported on a δ-neighbourhood of $\mathbb{S}^{n-1}$ normalized s.t. $\varphi_{\delta}(\xi) \sim 1$ near $|\xi| = 1$. Then the conjecture is

$$\left|\left| \int_{\mathbb{R}^n} \varphi_{\delta}(\xi) \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi_{L^p(\mathbb{R}^n)} \right|\right| \lesssim \delta^{-(n-1)((1/2-1/p)-1/p)} ||f||_{L^p(\mathbb{R}^n)} \quad \forall p > \frac{2n}{n-1}$$