Spherical Averages and the Spherical Maximal Function

Kevin Zhou

1 Spherical averages

Definition 1. Fix $n \geq 2$. For $t > 0$ and $x \in \mathbb{R}^n$ define the normalized *spherical average*

$$A_t f(x) \ := \ \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} f(x + t\omega)\, d\sigma(\omega), \tag{1}$$

where σ is surface measure on $\mathbb{S}^{n-1}$. The associated *spherical maximal operator* is

$$Mf(x) := \sup_{t>0} |A_t f(x)|.$$

Remark 1. For a general L^p function, the pointwise values of f on spheres are not a priori defined (values can change on sets of measure zero). Thus one first defines A_t on a dense class such as $C_c^\infty(\mathbb{R}^n)$ or $\mathcal{S}(\mathbb{R}^n)$ and then extends by continuity to L^p once uniform bounds are available.

A fundamental question is when the following the convergence holds:

$$\lim_{t \to 0^+} A_t f(x) = f(x) \qquad \text{for a.e. } x,$$

Theorem 1 (Stein; Bourgain)**.** *Let $Mf(x) := \sup_{t>0} |A_t f(x)|$. Then for $n \geq 2$ the operator M is bounded on $L^p(\mathbb{R}^n)$ if and only if*

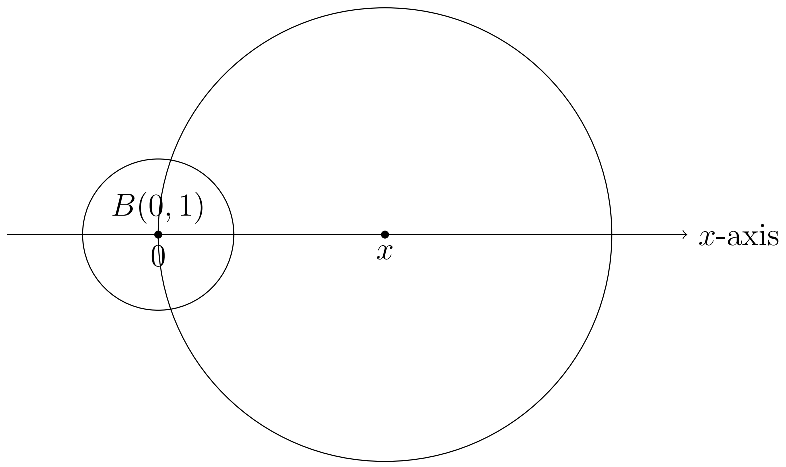
$$p > \frac{n}{n-1}.$$

Stein proved the sufficiency for $n \geq 3$; Bourgain settled the planar case $n = 2$. The necessity is fairly straightforward.

(Necessity) : Let $f = \mathbf{1}_{B(0,1)}$. For x with $|x| > 2$ there exists a radius $t \sim |x|$ for which the sphere $x + t\mathbb{S}^{n-1}$ cuts the unit ball in a spherical cap of σ-measure $\gtrsim |x|^{-(n-1)}$. Hence

$$Mf(x) \ \geq \ A_t f(x) \ \gtrsim \ |x|^{-(n-1)}.$$

Since $|x|^{-(n-1)} \notin L^p(\mathbb{R}^n)$ when $p \leq n/(n-1)$, boundedness fails at and below that endpoint.



2 Proof of sufficiency

The proof exploits two ideas: (i) a reduction to a *single scale* in t and a *single dyadic frequency* in ξ; (ii) oscillatory decay for the Fourier transform of surface measure.

We attempt to prove a simpler bound, which we currently claim without proof is sufficient:

$$\Big\| \sup_{t\in[1,2]} |A_t f| \Big\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

We utilise the Littlewood-Paley decomposition. Let $\{\beta_j\}_{j\in\mathbb{Z}}$ be a smooth frequency partition with $\widehat{\beta_j f}$ supported where $|\xi| \sim 2^j$. Note that it is enough to show that for some $\delta > 0$

$$\Big\| \sup_{t\in[1,2]} |A_t\, \beta_j f| \Big\|_{L^p(\mathbb{R}^n)} \lesssim 2^{-\delta j}\, \|\beta_j f\|_{L^p(\mathbb{R}^n)} \qquad \text{for all } j \in \mathbb{Z}, \tag{2}$$

whenever $p > \frac{n}{n-1}$. Summing the geometric series over j then yields the result we wish to prove.

To prove (2) let's view β_j as a convolution operator, and thus $A_t\beta_j(f)$ as also a convolution operator. To be more precise we write $A_t\beta_j(f) = A_t f * \beta_j(\text{kernel } \beta_j)(x)$ where the kernel of β_j satisfies

$$|\text{kernel } \beta_j| \lesssim \frac{1_{B(2^{-j})}}{|B(2^{-j})|}$$

We then have heuristically:

$$\left| \int_{w\in\mathbb{S}^{n-1}} (\text{kernel } \beta_j)(x+tw)d\sigma(w) \right| \lesssim 2^j 1_{||x|-t|<2^{-j}}$$

which gives us a new bound $\| \sup_{t\in[1,2]} |A_t\beta_j f|\|_{L^p} \lesssim 2^j \|f\|_{L^p}$. Unfortunately this growth in j makes the bound useless for proving (2). Thus we will need a better heuristic.

A better approach

We utilise the following heuristic (which is a lie):

$$|| \sup_{t\in[1,2]} |A_t\beta_j f|||_{L^p} \leq 2^{j/p} ||A_t\beta_j f||_{L^p(\mathbb{R}^n\times[1,2])}$$

Note that

$$A_t\beta_j f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi)\beta_j(\xi)e^{2\pi i(t|\xi|+x\cdot\xi}d\xi$$

and the coefficient of t in the complex exponential is $\simeq 2^j$ on the support of β_j. Thus as a function of t, $A_t\beta_j f(x)$ behaves like a function with frequency 2^j. As a result, there are three ways of "justifying" the lie above:

- (Sobolev embedding): We apply the Sobolev embedding theorem in the t variable (for a fixed x) to the function $t \mapsto 1_{[1,2]}(t)|A_t\beta_j f(x)|$. Thus heuristically, for every x, we have the approximation $2^{j/p}\|A_t\beta_j f(x)\|_{L^p([1,2])} \simeq \|A_t\beta_j f(x)\|_{W^{1/p,p}(\mathbb{R})}$. Thus

  $$|| \sup_{t\in[1,2]} |A_t\beta_j f|||_{L^p} \lesssim ||A_t\beta_j f||_{W^{1/p,p}} \quad p > 1$$

  However it is important to keep in mind that this is only heuristically justified using Sobolev embedding in the t variable.

- (Locally constant property): This refers to the idea that $A_t\beta_j f$ is approximately constant if t varies by 2^{-j} since the fourier transform of $t \mapsto A_t\beta_j f(x)$ is supported in $\{\tau : |\tau| \approx 2^j\}$.

  Let $g_t(x) = A_t\beta_j f(x)$. Choose $\phi \in C_c^\infty(\mathbb{R})$ with $\int_{\mathbb{R}^n} \phi = 1$. Let $\phi_j(s) = 2^j\phi(2^j s)$. Then $g_t = \phi_j * g_t$ and consider an interval I such that $|I| \approx 2^{-j}$. Let I^* be a slightly larger interval such that $I \subset I^* \subset [1/2, 4]$. Then Holder's inequality gives for any $r \in [1, \infty)$

  $$\sup_{t\in I} |g(t)| \leq (\phi_j * |g|)(t) \lesssim \left(2^j \int_{I^*} |g_t(s)|^r \, ds\right)^{1/r} \tag{3}$$

  Now we define $I_j = [a2^{-j}, (a + 1)2^{-j}]$. We take $r = p$ and then count the number of intervals I_j inside $[1, 2]$. Observing that this number is $O(1)$, we obtain the following inequality.

  $$\Big\| \sup_{t\in[1,2]} |g_t|\Big\|_{L^p(\mathbb{R}^n)} \lesssim 2^{j/p}\Big(\int_1^2 ||g_t||^p_{L^p} \, dt\Big)^{1/p} \tag{4}$$

- (Fundamental theorem of calculus) Suppose that $F(t)$ is smooth and supported on [1/2, 4], then we claim that

$$\sup_{t\in[1,2]} |F(t)| \lesssim ||F||_{L^p[1/2,4]}^{1-1/p} ||F'||_{L^p[1/2,4]}^{1/p}.$$

To see this, we first apply the fundamental theorem of calculus to see that

$$\begin{aligned} F(t)^p &= \int_{1/2}^t \partial_s(F^p) ds \\ &= \int_{1/2}^t pF^{p-1}F' ds \end{aligned}$$

Thus we have

$$|F(t)|^p \leq p \int_{1/2}^4 |F(s)|^{p-1} |F'(s)| ds$$

Now we pick a bump function $\eta \in C_c^\infty[1/2, 4]$ such that $\eta \equiv 1$ on $[1, 2]$. Then taking the supremum over t and then applying Holder's inequality in s gives the pointwise inequality:

$$\sup_{t\in[1,2]} |A_t\beta_j f(x)| \lesssim \|A_t\beta_j f(x)\|_{L^p([1/2,4])}^{1-\frac{1}{p}} \|\partial_t A_t \beta_j f(x)\|_{L^p([1/2,4])}^{\frac{1}{p}}.$$

Integrating in x and then applying Holder's in x gives:

$$\begin{aligned} || \sup_{t\in[1,2]} |A_t\beta_j f|||_{L^p} &\leq || \sup_{t\in[1/2,4]} \eta(t)|A_t\beta_j f(x)|||_{L^p_x L^p_t} \\ &\leq \|\eta(t)\, |A_t\beta_j f|\|_{L^p_x L^p_t}^{1-1/p} \left\|\partial_t\big(\eta(t)|A_t\beta_j f|\big)\right\|_{L^p_x L^p_t}^{1/p} \end{aligned}$$

Then since we chose our bump function η appropriately, the first term is approximately $\|\,|A_t\beta_j f|\|_{L^p}$ while the second term is bounded by $2^j||A_t\beta_j f||_{L^p}$. Thus we obtain the estimate

$$\begin{aligned} || \sup_{t\in[1,2]} |A_t\beta_j f|||_{L^p(\mathbb{R}^n)} &\leq 2^{j/p} ||||A_t\beta_j f||_{L^p(\mathbb{R}^n)}||_{L^p([1,2])} \\ &\leq 2^{j/p} \sup_{t\in[1,2]} ||A_t\beta_j f||_{L^p(\mathbb{R}^n)} \end{aligned}$$