Harmonic analysis

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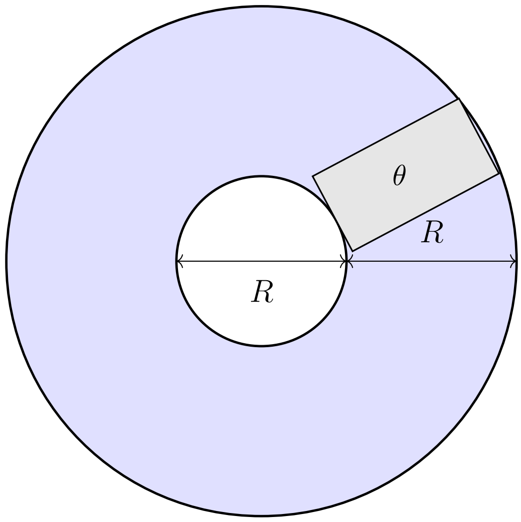
1 Decoupling for solutions of the half wave equations on $\mathbb{R}^{n+1}$



Suppose $f = \sum_\theta f_\theta$ where $supp \widehat{f_\theta} \subset \theta$. Now we define the half wave equation

$$u(x,t) = e^{it\sqrt{-\Delta}} f(x) = \sum_\theta u_\theta(x,t)$$

where $u_\theta(x,t) = e^{it\sqrt{-\Delta}} f_\theta(x)$.

Theorem

(Bourgain-Demeter Annals): If $p \geq 2$ and $R \geq 1$, and $u = \sum_\theta u_\theta$ is as above then

$$||u||_{L^p(\mathbb{R}^n \times [1,2])} \lessapprox R^{\alpha(p)} \left(\sum_\theta ||u_\theta||^p_{L^p(\mathbb{R}^n \times [1,2])} \right)^{1/p}$$

where

$$\alpha(p) = \begin{cases} \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{p}\right) & 2 \leq p \leq \frac{2(n+1)}{n-1} \\ (n-1)\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{p} & p \geq \frac{2(n+1)}{n-1} \end{cases}$$

It turns out to prove the above theorem suggested by Wolff, it is actually easier to prove a stronger theorem

Theorem

If $p \geq 2, R \geq 1, u = \sum_\theta u_\theta$ as above then

$$||u||_{L^p(\mathbb{R}^n \times [1,2])} \lessapprox R^{d(p)} \sum_\theta (||u_\theta||^2_{L^p(\mathbb{R}^n \times [1,2])})^{1/2}$$

where

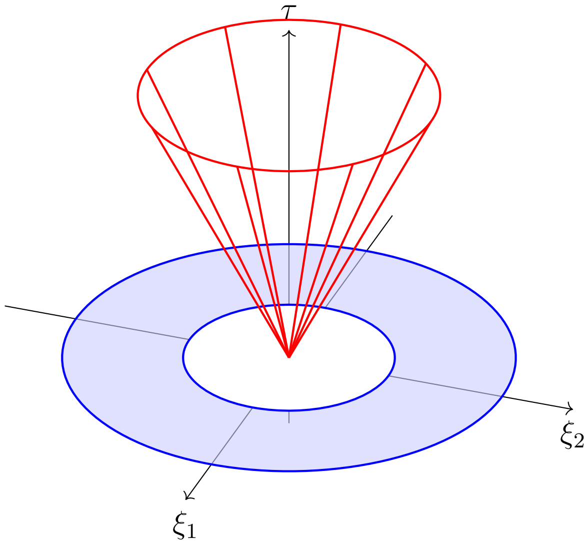
$$d(p) = \begin{cases} 0 & 2 \leq p \leq \frac{2(n+1)}{n-1} \\ \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{p} & p \geq \frac{2(n+1)}{n-1} \end{cases}$$

It is not difficult to derive the previous theorem from the above theorem using Holder's inequality. Another reduction done by Bourgain and Demeter in the same paper is that it turns out it has nothing to do with the wave equation.

Observation: *Supp* $\hat{u}_\theta \subset \{(\xi, |\xi| : \xi \in \theta)\}$, $u_\theta(x,t) = \int_{\mathbb{R}^n} e^{i(t|\xi| + x\cdot\xi)} \hat{f}_\theta(\xi) d\xi$
One should think of the above integrand as waves with frequencies sitting inside the light cone. As a result

$$Supp\ \widehat{u_\theta(x,t) 1_{[1,2]}}(t) \subset 1\text{-neighbourhood of } \{(\xi, |\xi| : \xi \in \theta)\}$$

$$(\xi, |\xi|) \subset \{(\xi, \tau) : \tau = |\xi|\}$$



One can see this as the fourier transform of the product is the convolution of each of the fourier transform. When you convolve, the supports add up. In light of the above fact, Bourgain and Demeter only had to prove:

Theorem

Suppose for all θ as before, u_θ is a Schwartz function in $\mathbb{R}^{n+1}$ such that $supp(\hat{u}_\theta) \subset$ 1-neighbourhood of $\{(\xi, |\xi|) : \xi \in \theta\}$. Then

$$\left|\left|\sum_\theta u_\theta\right|\right|_{L^p(\mathbb{R}^{n+1})} \lesssim R^{d(p)} \left(\sum_\theta ||u_\theta||^2_{L^p(\mathbb{R}^{n+1})}\right)^{1/2}$$

where $d(p)$ is as before.

Let us consider the cases where $d(p) = 0$.

Remark 1. *From* $2 \leq p \leq \frac{2(n+1)}{n-1}$ *we want roughly:*

$$\left|\left|\sum_\theta u_\theta\right|\right|_{L^p(\mathbb{R}^{n+1})} \lesssim \sum_\theta ||u_\theta||^2_{L^p(\mathbb{R}^{n+1})}$$

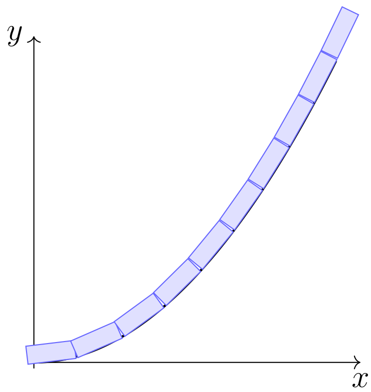
we have from the triangle inequality

$$\left|\left|\sum_\theta u_\theta\right|\right|_{L^p(\mathbb{R}^{n+1})} \leq \sum_\theta ||u_\theta||_{L^p} \leq (\#\theta)^{1/2} \left(\sum_\theta ||u_\theta||^2_{L^p}\right)^{1/2}$$

we need to beat the trivial bound by removing the constant term in front. For $p = 2$ this is true by orthogonality in L^2. We need to play a game of orthogonality in the other L^p spaces. It turns out we only need to prove the theorem at $p = \frac{2(n+1)}{n-1}$. There is a reduction to paraboloids in $\mathbb{R}^n$.

2 Parabola decoupling

Just for intuition, let's stick with the case where $n = 2$.



Theorem

Suppose $\{\theta\}$ is a collection of "disjoint" rectangles of dimensions $R^{-1/2} \times R^{-1}$ covering up a R^{-1} neighbourhood of the parabolda $\{(\xi, \xi^2) : \xi \in [0, 1]\}$. These θs are "tangent" to the parabola. Suppose $\forall\theta$ f_θ is a Schwartz function whose Fourier transform is supported in θ. Then for every $p \geq 2$ and for all $R \geq 1$

$$\left|\left|\sum_\theta f_\theta\right|\right|_{L^p(\mathbb{R}^2)} \lessapprox R^{d(p)} \left(\sum_\theta ||f_\theta||^2_{L^p(\mathbb{R}^2)}\right)^{1/2}$$

where $d(p)$ is as above with $n = 2$.

The relation to cone decoupling is from another paper from Pramanik-Seager (2009).

$$Cone = \{(\xi_1, \xi_2, \xi_3) : \xi_3^2 = \xi_1^2 + \xi_2^2\}$$

We change variables $\eta_1 = \xi_3 + \xi_1, \eta_3 = \xi_3 - \xi_1, \eta_2 = \xi_2$. Then the cone becomes

$$Cone = \{\eta_1 = \frac{\eta_2^2}{\eta_3}\}$$

Then we see that a neighbourhood of the cone is almost a cylinder over a neighbourhood of the parabola in $\mathbb{R}^2$.

Why is ℓ^2 decoupling easier than square function estimates?

The idea is called induction on scales. We cover a parabola with rectangles, then cover with bigger rectangles. We cover R^{-1} neighbourhood of parabolda by $\{\theta\}$ rectangles of size $R^{-1/2} \times R^{-1}$. We also cover S^{-1} neighbourhoods of parabola for $S^{-1} \geq R^{-1}$ by $\{\tau\}$ rectanges of size $S^{-1/2} \times S^{-1}$. If we induct on scale R, we may assume that $*$ bolds at scale $S < R$ and try to establish it for scale R. i.e. we assume

$$\left\|\sum_\theta f_\theta\right\|_{L^p(\mathbb{R}^2)} \leq S^{d(p)} \left(\sum_\tau \left\|\sum_{\theta\subset\tau} f_\theta\right\|_{L^p}^2\right)^{1/2}$$

It turns out that the induction hypothesis also almost says that for all τ,

$$\left\|\sum_{\theta\subset\tau} f_\theta\right\|_{L^p} \leq \left(\frac{R}{S}\right)^{d(p)} \left(\sum_{\theta\subset\tau} ||f_\theta||_{L^p}^2\right)^{1/2}$$

This would imply the theorem because then we would have

$$\left\|\sum_\theta f_\theta\right\|_{L^p} \leq S^{d(p)} \left(\frac{R}{S}\right)^{d(p)} \left(\sum_\tau \sum_{\theta\subset\tau} ||f_\theta||_{L^p}^2\right)^{1/2}$$

Unfortunately, the induction hypothesis assumes $\lessapprox$ and not a strict $\leq$ and we attempt to do something with that. By adjusting the induction hypothesis to use $\lessapprox$ instead, we get that

$$\left\|\sum_\theta f_\theta\right\|_{L^p} \lessapprox \left(\frac{R}{S}\right)^{d(p)} \left(\sum_\tau \sum_{\theta\subset\tau} ||f_\theta||_{L^p}^2\right)^{1/2}$$

Unfortunately this line of reasoning does not result in an actual proof as the constants implicit in the $\lessapprox$ would stack up in each induction step. Nevertheless, the closeness of this approach suggests that a similar approach may work.

Original proofs of Bourgain and Demeter uses induction on scales, bilinear/multilinear approach and multilinear Kakeya estimates. But we later realised there are simpler approaches.

3 Connection to discrete restriction estimates

Recall Tomas-Stein in $\mathbb{R}^n$: If $d\sigma$ is surface measure of paraboloid in $\mathbb{R}^n$ then the extension operator

$$||f(d\sigma)^{\vee}||_{L^{2(n+1)/(n-1)}(\mathbb{R}^n)} \leq C_n||f||_{L^2(paraboloid)}$$

Somehow this captures dispersion: waves travelling in different directions as

$$f(d\sigma)^{\vee} = \int_{paraboloid} f(\xi)e^{ix\cdot\xi}d\sigma(\xi)$$

Now consider a similar setup in a periodic setting. Is it true that

$$\left|\left|\sum_{k\in\mathbb{Z}^n, k_n=|k'|^2} a_k e^{2\pi i k\cdot x}\right|\right|_{L^{2(n+1)/(n-1)}([0,1]^n)} \leq C_n(\sum_{k\in\mathbb{Z}^n, k_n=|k'|^2} |a_k|^2)^{1/2}$$

It turns out this is **false**. e.g. $n = 2, a_k = 1$ if $k = \{1, 2, ..., N\}$ and 0 otherwise. But Bourgain and Demeter found out that

$$\left|\left|\sum_{k\in\mathbb{Z}^n, k_n=|k'|^2, |k|\leq N} a_k e^{2\pi i k\cdot x}\right|\right|_{L^{2(n+1)/(n-1)}([0,1]^n)} \leq C_\varepsilon N^\varepsilon(\sum_{k\in\mathbb{Z}^n, k_n=|k'|^2, |k|\leq N} |a_k|^2)^{1/2}$$

Turns out that ideas from exponential sum estimates also help give a simple proof of ℓ^2 decoupling for parabola in $\mathbb{R}^2$.