HARMONIC ANALYSIS LECTURE 2

LEKH BHATIA

1. Fourier Transform

We begin by introducing some heuristics to the Fourier transform, which are not literally true, but demonstrate some useful intuitions.

1-dimension

In 1-dimension the Fourier transform is given by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} \mathrm{d}x$$

which is usually easily computable. For example

$$\begin{aligned} \widehat{1_{[0,1]}}(\xi) &= \frac{e^{-2\pi i x \xi} - 1}{-2\pi i \xi} \\ &= e^{-2\pi i \frac{\xi}{2}} \left(\frac{\sin(2\pi\xi)}{2\pi\xi} \right) \end{aligned}$$

which decays like $\frac{1}{\xi}$ at ∞, and is approximately 1 at 0. This Fourier transform decays at infinity and has size roughly 1 in a unit interval around the origin. It also does not oscillate much on $[0,1]$. In this way, we think of this Fourier transform effectively as $1_{[0,1]}$ itself. Literally speaking, it would have worked if we had worked with the Gaussian function instead, but pretending that the characteristic over the unit interval is its own Fourier transform is a useful heuristics.

Now we consider an interval I of length r, and $\widehat{1_I}$. We consider two cases.

(1) Case 1: $0 \in I$. Then

$$\begin{aligned} \int_{\mathbb{R}} 1_{[0,r]} e^{-2\pi i x \xi} \mathrm{d}x &= \int_{\mathbb{R}} 1_{[0,1]} e^{-2\pi i x r \xi} r \mathrm{d}x \\ &= r 1_{[0,1]}(r\xi) \\ &= r 1_{[0,\frac{1}{r}]}(\xi) \end{aligned}$$

by change of variables in the first equality. This demonstrates that concentrated functions has a Fourier transform which is farther spread out. Sanity check: we can look at the L^2-norms of these functions and see that they are both $\sqrt{r}$. This indicates

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we are on the right track. Another check: The Fourier transform of an L^∞ normalized function should be L^1 normalized. This is the case here too.

(2) Now we consider $I = [a, a + r]$ with $a > 0$. Then

$$\begin{aligned}\int_a^{a+r} e^{-2\pi i x\xi}\mathrm{d}x &= \int_{\mathbb{R}} 1_{[0,r]}(x)e^{-2\pi i(x+a)\xi}\mathrm{d}x \\ &= e^{-2\pi i a\xi}\int_{\mathbb{R}} 1_{[0,r]}(x)e^{-2\pi i x\xi}\mathrm{d}x \\ &= e^{-2\pi i a\xi}\cdot r\cdot 1_{[0,\frac{1}{r}]}(\xi)\end{aligned}$$

We have simply added some oscillatory frequency a onto the Fourier transform from the previous case.

In summary, if $I \subset \mathbb{R}$ is an interval, we have heuristically that

$$\widehat{1_I}(\xi) = e^{ia\xi}|I|1_{[0,\frac{1}{|I|}]}(\xi)$$

for some (any) $a \in I$. If we are interested only in the modulus of this Fourier transform, then we can ignore the oscillation and say heuristically that $|\widehat{1_I}| = |I|1_{[0,\frac{1}{|I|}]}$.

Higher dimensions

Let us pick a rectangle θ. We can compute that

$$|\widehat{1_\theta}| \sim |\theta|1_{\theta^*}$$

where θ^* is the **dual** rectangle centered at 0. It is not hard to explicitly write down the dual. The dual of a rectangle of side lengths $r_1, \ldots, r_n$ is basically a rectangle with the same orientation centred at the origin, but with side lengths $r_1^{-1}, \ldots, r_n^{-1}$ instead. We can also write the transform itself

$$\widehat{1_\theta} \sim e^{i\omega\cdot\xi}|\theta|1_{\theta^*}$$

for $\omega \in \theta$. The plane waves in the Fourier transform oscillate in the direction of ω since these will correspond to the level sets of $\omega \cdot \xi = c$. This approximation holds more or less for all $\xi \in \theta^*$.

Given a rectangle $\theta \in \mathbb{R}^n$. We often tile $\mathbb{R}^n$ by translates of θ^*. Let Π_θ be this tiling. If $f \in C_c^\infty(\mathbb{R}^n)$ supported in θ. Once again we work by cases.

(1) Firstly if $0 \in \theta$

$$\widehat{f}(\xi) = \widehat{f1_\theta} = \hat{f} * \widehat{1_\theta}(\xi) \sim \hat{f} * |\theta|1_{\theta^*} \sim \frac{1}{|\theta^*|}\int_{\theta^*+\xi}\widehat{f}$$

where $*$ represents the convolution. In this case we see that the value of $\widehat{f}$ at ξ is simply given by the average value of $\widehat{f}$

at the tile $T \in \Pi_\theta$ which includes ξ. This is contingent on the equivalence $|\theta| = \frac{1}{|\theta^*|}$.

(2) Secondly if some $\omega \in \theta$. We define $f_\omega(x) = f(x+\omega)$. Then f_ω is supported in $\theta - \omega$, which does in fact include 0. Hence, $\widehat{f_\omega}$ is constant on any $T \in \Pi_\theta$, where we are effectively saying that

$$\widehat{f_\omega} = e^{2\pi i\omega\cdot\xi}\widehat{f}(\xi)$$

is constant on tiles, which implies that $|\widehat{f}|$ is also constant on tiles.

2. Sobolev Spaces

Definition 2.1. For $k \geq 0$ integer, $p \in [1,\infty]$, we define the Sobolev spaces $W^{k,p}(\mathbb{R}^n)$ as the completion of $C_c^\infty(\mathbb{R}^n)$ under the Sobolev norm

$$\|f\|_{k,p} = \left(\int_{\mathbb{R}^n} \sum_{|\alpha|\leq k} |\partial^\alpha f|^p \mathrm{d}x\right)^{\frac{1}{p}}.$$

We can also state a theorem.

Theorem 2.2 (Sobolev Embedding)**.** *If $kp > n$ then*

$$\|f\|_{L^\infty(\mathbb{R}^n)} \lesssim \|f\|_{W^{k,p}(\mathbb{R}^n)}.$$

Example 2.3 ($n = 1$, $k = 1$, $p = 1$)**.** For all $f \in W^{1,1}(\mathbb{R})$

$$\|f\|_{L^\infty(\mathbb{R})} \leq \|f'\|_{L^1(\mathbb{R})}$$

since

$$|f(x)| = \left|\int_{-\infty}^x f'(t)\mathrm{d}t\right| \leq \int_{-\infty}^x |f'(t)|\mathrm{d}t \leq \|f'\|_{L^1(\mathbb{R})}$$

Example 2.4 ($p = 2$)**.** We have for all $f \in W^{k,2}(\mathbb{R})$ that

$$\|f\|_{W^{k,2}(\mathbb{R}^n)} \approx \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^k |\hat{f}(\xi)|^2 \mathrm{d}\xi\right)^{\frac{1}{2}}$$

and hence

$$\|f\|_{L^\infty(\mathbb{R}^n)} \lesssim \|f\|_{W^{k,2}(\mathbb{R}^n)}$$

as long as $k > \frac{n}{2}$. This can be proven with Fourier inversion and Cauchy Schwarz.

Recall that

$$\partial^\alpha f = \int_{\mathbb{R}^n} (2\pi i\xi)^\alpha \hat{f}(\xi) e^{2\pi i x\cdot\xi}\mathrm{d}\xi,$$

a fundamental property of the Fourier transform. With this in mind, we would like to make the following definition (needs some care in practice since $|D|^\alpha$ is not defined for all $f \in L^p(\mathbb{R}^n)$).

Definition 2.5 (Fractional Order Sobolev Spaces)**.** For $\alpha \in [0,\infty)\setminus\mathbb{N}$,

$$|D|^\alpha f(x) = \int_{\mathbb{R}^n} |2\pi i\xi|^\alpha \hat{f}(\xi) e^{2\pi i x\cdot\xi} \mathrm{d}\xi$$

and we define the fractional order Sobolev spaces as follows

$$W^{\alpha,p}(\mathbb{R}^n) = \{f \in L^p(\mathbb{R}^n) : |D|^\alpha f \in L^p(\mathbb{R}^n)\}.$$

It is usually more convenient to use a Littlewood-Paley decomposition. Take a partition of unity adapted to dyadic annuli of radius 2^i for $i \in \mathbb{N}$. For instance, let η be the function which is 1 on $B(0,\frac{1}{2})$ and 0 outside $B(0,1)$. Polam: [Define $\eta_k(\xi) = \eta(2^{-k}x) - \eta(2^{1-k}x)$ for $k \geq 1$.] Then we define for $k \in \mathbb{N}$

$$P_0(f) = \mathcal{F}^{-1}(\hat{f}\cdot\eta)$$
$$P_k(f) = \mathcal{F}^{-1}(\hat{f}\cdot\eta_k).$$

This allows us to decompose f into a frequency-type decomposition: $\hat{f} = \sum_{k\geq 0}\widehat{P_k f}$ which indicates that $f = \sum_{k\geq 0} P_k f$

Theorem 2.6. *For $\alpha \geq 0$, $p \in (1,\infty)$*

$$\|f\|_{W^{\alpha,p}(\mathbb{R}^n)} \approx \left\| \left(\sum_{k\geq 0} |2^{k\alpha} P_k f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)}.$$

We have some heuristics for why this theorem makes sense to consider.

1. $D^\alpha P_k f \sim 2^{k\alpha} P_k f$
2. From $f = \sum_{k\geq 0} P_k f$ and we can derive the following norm bound

   $$\|f\|_{L^p(\mathbb{R}^n)} \leq \left\| \sum_{k\geq 0} |P_k f| \right\|_{L^p(\mathbb{R}^n)}.$$

   The theorem suggests that the correct bound is a square root cancellation bound instead.

Fact 2.7. *For $\alpha \geq 0$, $p \in (1,\infty)$*

$$\|P_k f\|_{W^{\alpha,p}(\mathbb{R}^n)} \approx 2^{k\alpha}\|P_k f\|_{L^p(\mathbb{R}^n)}.$$

This can be equivalently stated as follows. If $\hat{f}$ smooth is supported in an annulus where $|\xi| \approx R$ with $R > 1$, then

$$\|f\|_{W^{\alpha,p}(\mathbb{R}^n)} \approx R^\alpha \|f\|_{L^p(\mathbb{R}^n)}.$$

3. Wave Equation

Miyachi and Peral were able to resolve concentrations of waves in fixed time (with subsequent extensions by Seeger, Sogge, Stein). We will explain some of these in the next lecture. We may also try to understand the concentration in space-time, and this will lead to consideration related to the local smoothing phenomenon; today we will only have time to give a short statement (we will put this into context next time).

Definition 3.1 (Wave Equation)**.** We call the following boundary value problem the wave equation.

$$\begin{cases} \partial_t^2 u = \Delta_x u & (x, t \in \mathbb{R}^n \times \mathbb{R}) \\ u = f & t = 0 \\ \partial_t u = 0 & t = 0 \end{cases}$$

We can immediately write down solutions to the wave function in the following way

$$u_\pm(x,t) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i(\pm t|\xi| + x\cdot\xi)} \mathrm{d}\xi$$

with $u(x,t) = \frac{1}{2}(u_+(x,t) + u_-(x,t))$ being a solution.

Conjecture A (Sogge)**.** *Let* $n \geq 2$, $p = \frac{2n}{n-1}$ *and* u *be a solution to the wave equation with initial data* $f \in \mathcal{S}(\mathbb{R}^n)$ *as above. If* $\hat{f}$ *is supported in an annulus of size* $R > 1$ *and* $\epsilon > 0$ *then*

$$\|u(x,t)\|_{L^p(\mathbb{R}^n \times [1,2])} \lesssim_\epsilon R^\epsilon \|f\|_{L^p(\mathbb{R}^n)}. \tag{1}$$

This was recently solved by Guth, Wang and Zhang in 2020 for $n = 2$. The cases $n \geq 3$ are wide open.