LOCAL SMOOTHING IMPLIES SPHERICAL BOCHNER-RIESZ

PO-LAM YUNG

Sogge [1] observed that the local smoothing conjecture he introduced there implies Bochner-Riesz for the sphere. Below we give some details of that argument. (A variant of this argument, based on a dyadic decomposition of the Bochner-Riesz multiplier, is given in Proposition 3.2 of [2].)

1. Computation of the Fourier transform of $t^{-(1+\delta)}$

First, for any $\delta \in (0,1)$, one can define a tempered distribution $t^{-(1+\delta)}$ on $\mathbb{R}$ by

$$f \in \mathcal{S}(\mathbb{R}) \mapsto \int_{\mathbb{R}} \frac{f(t)-f(0)}{t^{1+\delta}} dt.$$

(Here we need to choose a branch of log to define fractional powers of t when $t<0$; let's choose the branch of log with branch cut along the negative imaginary axis from now on.) Sometimes this tempered distribution is written as $(t+i0^+)^{-(1+\delta)}$, because for $f \in \mathcal{S}(\mathbb{R})$, the above is also equal to

$$\lim_{\epsilon \to 0^+} \int_{\mathbb{R}} \frac{f(t)}{(t+i\epsilon)^{1+\delta}} dt.$$

(Note that $\int_{\mathbb{R}} \frac{1}{(t+i\epsilon)^{1+\delta}} dt = 0$ for every $\epsilon \neq 0$, and hence $\int_{\mathbb{R}} \frac{f(t)}{(t+i\epsilon)^{1+\delta}} dt = \int_{\mathbb{R}} \frac{f(t)-f(0)}{(t+i\epsilon)^{1+\delta}} dt$; we can then let $\epsilon \to 0^+$ using dominated convergence theorem if $f \in \mathcal{S}(\mathbb{R})$.)

One can compute the Fourier transform of the tempered distribution $t^{-(1+\delta)}$, by noting the pointwise equality

$$\int_{\mathbb{R}} \frac{e^{-it\tau}-1}{t^{1+\delta}} dt = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}} \frac{e^{-it\tau}}{(t+i\epsilon)^{1+\delta}} dt = c_\delta \tau_+^\delta$$

where $c_\delta := -\frac{2\pi i e^{-i\pi\delta/2}}{\delta\Gamma(\delta)} \neq 0$ and $\tau_+ = \max\{\tau, 0\}$. In fact,

$$\int_{\mathbb{R}} \frac{e^{-it\tau}}{(t+i\epsilon)^{1+\delta}} dt = \int_{-R}^{R} \frac{e^{-it\tau}}{(t+i\epsilon)^{1+\delta}} dt + O(R^{-\delta})$$

This shows

$$\int_{\mathbb{R}} \frac{e^{-it\tau}}{(t+i\epsilon)^{1+\delta}} dt = e^{-\epsilon\tau} \int_{\Gamma_R^+} \frac{e^{-iz\tau}}{z^{1+\delta}} dz + O(\epsilon R^{-(1+\delta)}) + O(R^{-\delta})$$

where Γ_R^+ is the upper half circle of radius R centered at the origin, going from $-R$ to R. If $\tau \leq 0$, then $|e^{-iz\tau}| \leq 1$ along Γ_R^+, so

$$\int_{\Gamma_R^+} \frac{e^{-iz\tau}}{z^{1+\delta}} dz = O(R^{-\delta})$$

as well, and hence

$$\lim_{\epsilon \to 0^+} \int_{\mathbb{R}} \frac{e^{-it\tau}}{(t+i\epsilon)^{1+\delta}} dt = 0.$$

If $\tau > 0$, we use instead

$$\begin{aligned}\int_{\mathbb{R}} \frac{e^{-it\tau}}{(t+i\epsilon)^{1+\delta}}dt &= \int_{-R}^{R} \frac{e^{-it\tau}}{(t+i\epsilon)^{1+\delta}}dt + O(R^{-\delta}) \\ &= -\frac{1}{\delta}\int_{-R}^{R} e^{-it\tau}\frac{d}{dt}\frac{1}{(t+i\epsilon)^{\delta}}dt \\ &= -\frac{i\tau}{\delta}\int_{-R}^{R}\frac{e^{-it\tau}}{(t+i\epsilon)^{\delta}}dt + O(R^{-\delta}) \\ &= -\frac{i\tau}{\delta}e^{-\epsilon\tau}\int_{-R+i\epsilon}^{R+i\epsilon}\frac{e^{-iz\tau}}{z^{\delta}}dz + O(R^{-\delta}) \\ &= -\frac{i\tau}{\delta}e^{-\epsilon\tau}\Big(\int_{-R}^{-\epsilon} + \int_{\epsilon}^{R}\frac{e^{-it\tau}}{t^{\delta}}dt + \int_{\Gamma_\epsilon}\frac{e^{-iz\tau}}{z^{\delta}}dz\Big) + O(R^{-\delta}) \\ &= -\frac{i\tau}{\delta}e^{-\epsilon\tau}\int_{\epsilon}^{R}\frac{e^{-it\tau}+e^{it\tau}e^{-i\pi\delta}}{t^{\delta}}dt + O(\epsilon^{1-\delta}) + O(R^{-\delta}).\end{aligned}$$

For $0<\delta<1$,

$$\int_0^R t^{-\delta}e^{-it}dt = -ie^{i\pi\delta/2}\int_0^{Ri} z^{-\delta}e^{-z}dz = -ie^{i\pi\delta/2}\Big(\int_0^R t^{-\delta}e^{-t}dt + \int_{C_R} z^{-\delta}e^{-z}dz\Big)$$

where C_R is the quarter circle of radius R from R to Ri. But $\int_0^\infty t^{-\delta}e^{-t}dt = \Gamma(1-\delta)$, while

$$\Big|\int_{C_R} z^{-\delta}e^{-z}dz\Big| \leq \int_0^{\pi/2} R^{1-\delta}e^{-R\cos\theta}d\theta \leq R^{1-\delta}\int_0^{\pi/2} e^{-R(1-\frac{2\theta}{\pi})}d\theta = R^{1-\delta}e^{-R}\frac{\pi}{2R}(e^R-1) \to 0$$

as $R\to+\infty$. Hence we have

$$\lim_{R\to+\infty}\int_0^R t^{-\delta}e^{-it}dt = -ie^{i\pi\delta/2}\Gamma(1-\delta).$$

Letting $R\to+\infty$ and then $\epsilon\to 0^+$ in our earlier display, we have

$$\lim_{\epsilon\to0^+}\int_{\mathbb{R}}\frac{e^{-it\tau}}{(t+i\epsilon)^{1+\delta}}dt = -\frac{i\tau}{\delta}\tau^{\delta-1}(-ie^{i\pi\delta/2}+ie^{-i\pi\delta/2}e^{-i\pi\delta})\Gamma(1-\delta) = c_\delta\tau^\delta$$

where

$$c_\delta = -\frac{2i\sin(\pi\delta)e^{-i\pi\delta/2}\Gamma(1-\delta)}{\delta} = -\frac{2\pi ie^{-i\pi\delta/2}}{\delta\Gamma(\delta)} \neq 0 \quad \text{for } 0<\delta<1.$$

2. Bounding the Bochner-Riesz operator

Let $0<\delta<1$ and S^δ be the Bochner-Riesz multiplier $(1-|\xi|^2)_+^\delta$. We can write

$$(1-|\xi|^2)_+^\delta = \phi(\xi)(1-|\xi|^2)^\delta + \eta(\xi)(1-\phi(\xi))(1+|\xi|)^\delta(1-|\xi|)_+^\delta$$

where ϕ is a C^∞ function supported in the ball $|\xi|\leq 1/2$, and $\eta(\xi)$ is another C^∞ function with compact support that is identically equal to 1 in the ball $|\xi|\leq 1$. This decomposes

$$S^\delta = S_0^\delta + S_1^\delta\underline{S}^\delta$$

where S_0^δ is given by the Fourier multiplier $\phi(\xi)(1-|\xi|^2)^\delta$, S_1^δ is given by the Fourier multiplier $\eta(\xi)(1-\phi(\xi))(1+|\xi|)^\delta$, and $\underline{S}^\delta$ is given by the Fourier multiplier $(1-|\xi|)_+^\delta$. Since S_0^δ and S_1^δ are given by convolutions against Schwartz functions, they are bounded on $L^p(\mathbb{R}^n)$ for any $p\geq 1$. Hence the boundedness of the Bochner-Riesz multiplier is the same as the boundedness of $\underline{S}^\delta$.

Now writing $u(x,t) = e^{it\sqrt{-\Delta}}f(x)$, we have

$$\underline{S}^\delta f(x) = \frac{1}{c_\delta}\int_{\mathbb{R}} \frac{e^{-it}u(x,t) - f(x)}{t^{1+\delta}}dt$$

and hence

$$\|\underline{S}^\delta f\|_{L^p} \leq \frac{1}{c_\delta}\int_{\mathbb{R}} \Big\|\frac{e^{-it}u(x,t) - f(x)}{t^{1+\delta}}\Big\|_{L^p_x}dt$$

For $|t| \leq 1$, we write

$$e^{-it}u(x,t) - 1 = e^{-it}(u(x,t) - f(x)) + (e^{-it} - 1)f(x),$$

and use

$$\Big\|\frac{u(x,t) - f(x)}{t}\Big\|_{L^p_x} \lesssim \|f\|_{L^p}$$

uniformly in t, when f has Fourier support inside the unit ball. In fact, if $\eta(\xi)$ is a compactly supported smooth function such that $\eta = 1$ on the unit ball, then the kernel for the multiplier operator $\eta(\xi)\frac{e^{it|\xi|}-1}{t}$ is

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \eta(\xi)\frac{e^{it|\xi|} - 1}{t}e^{ix\cdot\xi}d\xi = O(1 + |x|)^{-n-1}$$

by splitting the integral according to whether $|\xi| \geq |x|^{-1}$ or not, and repeated integration by parts in ξ. Hence for $|t| \leq 1$,

$$\Big\|\frac{e^{-it}u(x,t) - f(x)}{t^{1+\delta}}\Big\|_{L^p_x} \lesssim t^{-\delta}\|f\|_{L^p}$$

for all $p \in [1, \infty]$.

For $|t| \simeq 2^k$, $k \geq 1$, we let $u_k(x,t) := 2^{kn/p}u(2^kx, 2^kt)$ and $f_k(x) = 2^{kn/p}f(2^kx)$ so that $u_k(x,t) = e^{it\sqrt{-\Delta}}f_k(x)$ and

$$\int_{|t|\simeq 2^k} \frac{\|u(x,t)\|_{L^p(\mathbb{R}^n)}}{t^{1+\delta}}dt \lesssim 2^{-k\delta}\Big(\int_{|t|\simeq 2^k} \|u(x,t)\|^p_{L^p(\mathbb{R}^n)}\frac{dt}{t}\Big)^{1/p} \simeq 2^{-k\delta}\|u_k\|_{L^p_{|t|\simeq 1}L^p_x}.$$

We may assume f has Fourier support inside the ball of radius 2 centered at 0, so that $\widehat{f_k}$ is supported on a ball of radius $\simeq 2^k$. Let $p = \frac{2n}{n-1}$ from now on. Local smoothing gives

$$\|u_k\|_{L^p_{|t|\simeq 1}L^p_x} \lesssim_\varepsilon 2^{k\epsilon}\|f_k\|_{L^p} = 2^{k\epsilon}\|f\|_{L^p}.$$

We choose $\varepsilon = \delta/2 > 0$. Then we may sum over k, and obtain

$$\int_{|t|\geq 1} \frac{\|u(x,t)\|_{L^p_x}}{t^{1+\delta}}dt \lesssim \|f\|_{L^p}.$$

This proves

$$\int_{|t|\geq 1} \Big\|\frac{e^{-it}u(x,t) - f(x)}{t^{1+\delta}}\Big\|_{L^p_x}dt \lesssim \|f\|_{L^p}$$

for $\delta > 0$ and $p = \frac{2n}{n-1}$. Hence S^δ is bounded on $L^p(\mathbb{R}^n)$ for all $\delta > 0$.

References

[1] Christopher D. Sogge, *Propagation of singularities and maximal functions in the plane*, Inventiones Mathematicae **104** (1991), no. 2, 349–376, available at https://eudml.org/doc/143887.Zbl 0754.35004

[2] David Beltran, Jonathan Hickman, and Christopher D. Sogge, *Sharp local smoothing estimates for Fourier integral operators*, Geometric aspects of harmonic analysis. Proceedings of the INdAM meeting, Cortona, Italy, June 25–29, 2018, Cham: Springer, 2021, pp. 29–105, DOI 10.1007/978-3-030-72058-2_2, available at www.pure.ed.ac.uk/ws/files/112362823/1812.11616.pdf.