HARMONIC ANALYSIS LECTURE 5

NOAH GORRELL

Abstract. In this lecture, we finish a proof sketch of a weakening of Stein's spherical maximal theorem, first in dimension $n \geq 3$, and then for dimension $n = 2$. Finally, we state two significant consequences of the local smoothing conjecture.

Let us begin by reviewing the problem at hand.

For each $t > 0$, we define $\mathrm{d}\sigma_t$ to be the normalised surface measure on $tS^{n-1} \subset \mathbb{R}^n$, the sphere of radius t centred at 0, i.e. $\int_{tS^{n-1}} \mathrm{d}\sigma_t = 1$.

Following Stein (1976), we consider functions $f \in \mathscr{S}(\mathbb{R}^n)$. For any $x \in \mathbb{R}^n$, the convolution $f * \mathrm{d}\sigma_t(x) = \int_{tS^{n-1}} f(x-y)\,\mathrm{d}\sigma_t(y)$ computes the average value of f on the sphere of radius t centred at x.[1]

Defining, $Mf(x) = \sup_{t>0} |f * \mathrm{d}\sigma_t(x)|$, Stein (1976) proved the following result.

Theorem 1 (Stein, 1976)**.** Suppose $n \geq 3$. Then, we have $\|Mf\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p}$ for all $f \in \mathscr{S}(\mathbb{R}^n)$ if and only if $p > \frac{n}{n-1}$.

This result was extended to dimension $n = 2$ by Bourgain (1985), and a simplified proof was produced by Mockenhaupt, Seeger, and Sogge (1992). In the previous lecture we considered the following slight weakening of this result.

Theorem 2. Suppose $n \geq 2$. We have $\| \sup_{t\in[1,2]} |f * \mathrm{d}\sigma_t| \|_{L^p} \lesssim \|f\|_{L^p}$ if and only if $p > \frac{n}{n-1}$.

We observed that the condition $p > \frac{n}{n-1}$ was necessary and began a sketch of its sufficiency. Today's goal is to complete this proof sketch.

Preliminaries

Our first step was to decompose f as $P_0 f + \sum_{k=1}^{\infty} P_k f$ via Littlewood-Paley decomposition. Recall that $P_0 f$ has Fourier support in a ball near zero, and $P_k f$ has Fourier support in an annulus $\{|\xi| \sim 2^k\}$.

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[1] One encounters a problem when trying to define such an average for general $f \in L^p(\mathbb{R}^n)$: namely, such functions are almost-everywhere defined and the sphere has measure zero in $\mathbb{R}^n$.

By the triangle inequality, it suffices to obtain a uniform bound on $\| \sup_{t \in [1,2]} |f * \mathrm{d}\sigma_t| \|_{L^p}$ for all $k \geq 0$.

In Lecture 4, we obtained an easy bound, by comparing with the Hardy-Littlewood maximal function. Namely, for every $p > 1$,

$$\| \sup_{t \in [1,2]} |P_k f * \mathrm{d}\sigma_t| \|_{L^p} \leq 2^k \|f\|_{L^p}. \tag{1}$$

As $k \to \infty$, this bound worsens quickly. However, this at least shows that the map $f \in \mathscr{S}(\mathbb{R}^n) \mapsto \sup_{t \in [1,2]} |P_k f * \mathrm{d}\sigma_t|$ is L^p-bounded for every k. In particular, we have a bound for $k = 0$, and so we just need to obtain suitable bounds for $k \geq 1$.

Insights about $\widehat{\mathrm{d}\sigma_t}$. Let $\mathrm{d}\sigma \coloneqq \mathrm{d}\sigma_1$. We have the following fact:

$$\widehat{\mathrm{d}\sigma}(\xi) = a_+(\xi) e^{2\pi i |\xi|} + a_-(\xi) e^{-2\pi i |\xi|}, \tag{2}$$

where $|\partial_\xi^\alpha a_\pm(\xi)| \lesssim_\alpha (1 + |\xi|)^{-\frac{n-1}{2} - |\alpha|}$ for every multi-index α (see Chapter VIII in Stein's Harmonic Analysis).

Also, by a change-of-variables, one sees that $\widehat{\mathrm{d}\sigma_t}(\xi) = \widehat{\mathrm{d}\sigma}(t\xi)$. Therefore, we have $|\widehat{\mathrm{d}\sigma_t}(\xi)| \lesssim |t\xi|^{-\frac{n-1}{2}}$. In fact, since we are only taking $t \in [1,2]$, we simply have $|\widehat{\mathrm{d}\sigma_t}(\xi)| \lesssim |\xi|^{-\frac{n-1}{2}}$, with the implicit constant independent of t. We note that this bound improves as $n \to \infty$. (The geometric reason for the exponent $-\frac{n-1}{2}$ is that it counts the number of principal curvatures on the sphere.) In particular, on the Fourier support of $P_k f$, we have $|\widehat{\mathrm{d}\sigma_t}(\xi)| \lesssim (2^k)^{-\frac{n-1}{2}}$.

Moreover, because of (2), we think of $\widehat{\mathrm{d}\sigma} \approx e^{2\pi i |\xi|} |\xi|^{-\frac{n-1}{2}}$. Of course, this is only a heuristic but, via Fourier inversion, it gives us the following calculation:

$$P_k f * \mathrm{d}\sigma(x) \approx \int_{|\xi| \sim 2^k} \hat{f}(\xi) |\xi|^{-\frac{n-1}{2}} e^{2\pi i (|\xi| + x \cdot \xi)} \, \mathrm{d}\xi \tag{3}$$

Last time, we also made arguments (for example, by Sobolev embedding) that the following heuristic was reasonable, although not strictly correct:

$$\Big\| \sup_{t \in [1,2]} |P_k f * \mathrm{d}\sigma_t| \Big\|_{L^p(\mathbb{R}^n)} \lesssim 2^{k/p} \|P_k f * \mathrm{d}\sigma_t\|_{L^p(\mathbb{R}^n_x \times [1,2]_t)} \tag{4}$$

By Fubini's theorem, we can immediately bound the right-hand side by $2^{k/p} \sup_{t \in [1,2]} \|P_k f * \mathrm{d}\sigma_t\|_{L^p(\mathbb{R}^n)}$.

Let us apply this bound to $p = 2$. By Plancherel's theorem, we have $\|P_k f * \mathrm{d}\sigma_t\|_{L^2(\mathbb{R}^n)} = \|\widehat{P_k f * \mathrm{d}\sigma_t}\|_{L^2(\mathbb{R}^n)}$, and since the Fourier transform interchanges convolutions with products, we have

$$\begin{aligned}\|\sup_{t\in[1,2]}|P_kf*\mathrm{d}\sigma_t|\|_{L^2(\mathbb{R}^n)} &\leq 2^{k/2}\sup_{t\in[1,2]}\|P_kf*\mathrm{d}\sigma_t\|_{L^2(\mathbb{R}^n)}\\ &= 2^{k/2}\sup_{t\in[1,2]}\|\widehat{P_kf}\,\widehat{\mathrm{d}\sigma_t}\|_{L^2(\mathbb{R}^n)}\\ &\leq 2^{k/2}\sup_{t\in[1,2]}\|\widehat{P_kf}\|_{L^2(\mathbb{R}^n)}\|\widehat{\mathrm{d}\sigma_t}\|_{L^\infty(\mathrm{supp}(\widehat{P_kf}))}\\ &\lesssim 2^{k/2}\|\widehat{P_kf}\|_{L^2(\mathbb{R}^n)}(2^k)^{-\frac{n-1}{2}}\\ &= 2^{-\frac{k(n-2)}{2}}\|\widehat{P_kf}\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

The third inequality holds because $\mathrm{supp}(\widehat{P_kf})=\{|\xi|\sim 2^k\}$.

Finally, since $\|\widehat{P_kf}\|_{L^2(\mathbb{R}^n)}\leq\|\hat{f}\|_{L^2(\mathbb{R}^n)}=\|f\|_{L^2(\mathbb{R}^n)}$, we obtain the bound

$$\|\sup_{t\in[1,2]}|P_kf*\mathrm{d}\sigma_t|\|_{L^2(\mathbb{R}^n)}\lesssim 2^{-\frac{k(n-2)}{2}}\|f\|_{L^2(\mathbb{R}^n)}. \tag{5}$$

STEIN'S ARGUMENT FOR $n\geq 3$

When $n\geq 3$, the scaling term $2^{-\frac{k(n-2)}{2}}$ in (5) decays exponentially as $k\to\infty$, so this establishes the forward direction of Theorem 2 for $p=2$ and $n\geq 3$. Via interpolation, let us now establish the result for all $\frac{n}{n-1}<p<2$.

Fix $\frac{n}{n-1}<p<2$ and let $1<p_0<p$. We combine the easy bound in (1) for $L^{p_0}(\mathbb{R}^n)$ with the bound in (5) for $L^2(\mathbb{R}^n)$. Choose θ so that $\frac{1}{p}=\frac{\theta}{p_0}+\frac{1-\theta}{2}$ (i.e., $\theta=\frac{1/p-1/2}{1/p_0-1/2}$). Since the mapping $f\mapsto\sup_{t\in[1,2]}|P_kf*\mathrm{d}\sigma_t|$ is not linear, the Riesz-Thorin theorem does not immediately apply. We can remedy this situation by 'linearising the operator'. In short, for each continuous function f, we construct a linear operator T such that $Tf=\sup_{t\in[1,2]}|P_kf*\mathrm{d}\sigma_t|$ and T has the same L^{p_0} and L^2 bounds as the spherical maximal operator. Then, we apply the Riesz-Thorin theorem to T. Since $\|T\|_{p_0}\leq 2^k$ and $\|T\|_2\leq(2^k)^{-\frac{n}{2}+1}$, we have

$$\|T\|_p\leq(2^k)^\theta(2^k)^{(1-\theta)(-\frac{n}{2}+1)}=(2^k)^{-\frac{n}{2}+1+\frac{n\theta}{2}}.$$

As $p_0\searrow 1$, we have $\theta\to\frac{2}{p}-1$. Therefore,

$$\|T\|_p\leq(2^k)^{-\frac{n}{2}+1+\frac{n}{p}-\frac{n}{2}}=(2^k)^{1-n+\frac{n}{p}}.$$

Thus, for every $f\in\mathscr{S}(\mathbb{R}^n)$, we have

$$\|\sup_{t\in[1,2]}|P_kf*\mathrm{d}\sigma_t|\|_{L^p}\leq(2^k)^{1-n+\frac{n}{p}}\|f\|_{L^p}.$$

By the triangle inequality, we conclude that

$$\|\sup_{t\in[1,2]}|f*\mathrm{d}\sigma_t|\|_{L^p}\leq\sum_{k=1}^{\infty}(2^k)^{1-n+\frac{n}{p}}\|f\|_{L^p}.$$

This bound is finite if and only if $1-n+\frac{n}{p}<0$, and this occurs precisely when $p>\frac{n}{n-1}$.

This establishes Theorem 2 for $\frac{n}{n-1}<p\leq 2$ and $n\geq 3$.

The interpolation argument for $2<p<\infty$ is even easier. We note that $\|\sup_{t\in[1,2]}|f*\mathrm{d}\sigma_t|\|_{L^\infty}\leq\|f\|_{L^\infty}$, since a spherical average at every point is necessarily bounded by $\sup_{x\in\mathbb{R}^n}|f(x)|$. Since we also have boundedness on $L^2(\mathbb{R}^n)$, we again linearise the operator to see that the spherical maximal operator is bounded on $L^p(\mathbb{R}^n)$ for all $2<p<\infty$.

Mockenhaupt, Seeger, and Sogge's Argument for $n=2$

When $n=2$, we wish to prove the spherical maximal operator is bounded on L^p for $p>2$. Let us consider once more the heuristic (3). Since we are considering only $t\in[1,2]$ and $n=2$, we have

$$\begin{aligned}P_kf*\mathrm{d}\sigma_t(x)&\approx\int_{|\xi|\sim 2^k}\hat{f}(\xi)|t\xi|^{-1/2}e^{2\pi i(t|\xi|+x\cdot\xi)}\,\mathrm{d}\xi\\&\approx(2^k)^{-1/2}\int_{|\xi|\sim 2^k}\hat{f}(\xi)e^{2\pi i(t|\xi|+x\cdot\xi)}\,\mathrm{d}\xi\\&=2^{-k/2}e^{it\sqrt{-\Delta}}f(x).\end{aligned}$$

Therefore, by (4),

$$\|\sup_{t\in[1,2]}|P_kf*\mathrm{d}\sigma_t|\|_{L^p(\mathbb{R}^n)}\lesssim 2^{k/p}2^{-k/2}\|e^{it\sqrt{-\Delta}}f\|_{L^p(\mathbb{R}^2_x\times[1,2]_t)}. \tag{6}$$

The local smoothing conjecture for $n=2$ (Guth, Wang, & Zhang, 2020)[2] determines that when $p_0>4$, we have

$$\|e^{it\sqrt{-\Delta}}f\|_{L^{p_0}(\mathbb{R}^2_x\times[1,2]_t)}\lesssim(2^k)^{\frac{1}{2}-\frac{1}{p_0}-\frac{1}{p_0}}\|f\|_{L^{p_0}(\mathbb{R}^2)}.$$

Moreover, conservation of energy says that

$$\|e^{it\sqrt{-\Delta}}f\|_{L^2(\mathbb{R}^2_x\times[1,2]_t)}=(2^k)^{\frac{1}{2}-\frac{1}{2}-0}\|f\|_{L^2(\mathbb{R}^2)}.$$

Interpolating these results, we see that for any $2<p<p_0$, there exists a number $\epsilon(p)>0$ such that

$$\|e^{it\sqrt{-\Delta}}f\|_{L^p(\mathbb{R}^2_x\times[1,2]_t)}\lesssim(2^k)^{\frac{1}{2}-\frac{1}{p}-\epsilon(p)}\|f\|_{L^p(\mathbb{R}^2)}.$$

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[2] In fact, there is a partial result for local smoothing that was known in 1992 when Mockenhaupt, Seeger, and Sogge published their proof and suffices for this purpose.

Inserting this bound into (6), we get

$$\begin{aligned}\| \sup_{t\in[1,2]} |P_k f * \mathrm{d}\sigma_t|\|_{L^p(\mathbb{R}^n)} &\lesssim 2^{k/p}2^{-k/2}(2^k)^{\frac{1}{2}-\frac{1}{p}-\epsilon(p)}\|f\|_{L^p(\mathbb{R}^2)}\\ &= 2^{-\epsilon(p)k}\|f\|_{L^p(\mathbb{R}^2)}.\end{aligned}$$

Since this bound decays exponentially as $k \to \infty$, this determines a bound on $\| \sup_{t\in[1,2]} |f * \mathrm{d}\sigma_t|\|_{L^p(\mathbb{R}^n)}$ and concludes the proof for $n = 2$.

Consequences of Local Smoothing

Should the local smoothing conjecture hold, one could apply the above approach to $n \geq 3$. Hence, we may view Stein's spherical maximal theorem as a consequence to the local smoothing conjecture. (We stress however that the spherical maximal theorem has been proven, without a need for local smoothing.) Earlier in the course, we also saw that the local smoothing conjecture implies the Bochner-Riesz conjecture. Here, we state two other conjectures that follow from the Bochner-Riesz conjecture.

Restriction Conjecture for the sphere. For any $f \in L^1(S^{n-1})$, define a function Ef on $\mathbb{R}^n$ by $Ef(x) = \int_{S^{n-1}} f(\xi)e^{2\pi i x\cdot\xi}\,\mathrm{d}\xi$. The restriction conjecture states that E is a bounded mapping $L^p(S^{n-1}) \to L^p(\mathbb{R}^n)$ for all $p > \frac{2n}{n-1}$. The dual (hence, equivalent) statement is that the map[3] $f \mapsto \hat{f}|_{S^{n-1}}$ is a bounded mapping $L^p(\mathbb{R}^n) \to L^p(S^{n-1})$ for all $p < \frac{2n}{n+1}$. In fact, by an interpolation argument, it follows that this is a bounded mapping $L^p(\mathbb{R}^n) \to L^q(S^{n-1})$ for all $p < \frac{2n}{n+1}$ and $q \leq \frac{n-1}{n+1}p'$. Indeed, this is how the conjecture is stated in Tao's blog post on the subject, and it is known that for (p, q) outside of this range, the restriction map is not bounded $L^p(\mathbb{R}^n) \to L^q(S^{n-1})$.

(Maximal) Kakeya Conjecture. A δ-tube in $\mathbb{R}^n$ is an n-dimensional cylinder with height 1 and radius δ. Suppose $\mathbf{T}$ is a family of δ-tubes in $\mathbb{R}^n$ pointing in a δ-separate set of directions. Then for $\epsilon > 0$ and any family of non-negative constants $\{c_T\}_{T\in\mathbf{T}}$ we have

$$\left\|\sum_{T\in\mathbf{T}} c_T \mathbf{1}_T\right\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \lesssim_\epsilon \delta^{-\epsilon}\left(\sum_{T\in\mathbf{T}} c_T^{\frac{n}{n-1}}|T|\right)^{\frac{n-1}{n}}.$$

(Note the implicit constant here does not depend on the family $\{c_T\}_{T\in\mathbf{T}}$.)

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[3] As noted previously, it takes some work to define this map properly, since the values of $\hat{f}$ may not be defined on the sphere.

It turns out that the Bochner-Riesz conjecture implies the restriction conjecture (Tao, 1999) and the restriction conjecture implies the Kakeya conjecture (see these notes by Oliveira).