# HARMONIC ANALYSIS LECTURE 8

NOAH GORRELL

Recall the setting for the local smoothing conjecture. We assume $f \in \mathscr{S}(\mathbb{R}^n)$ has Fourier support in an annulus $\{|\xi| \sim R\}$, we decompose $f = \sum_\theta f_\theta$ into finitely many pieces, where each $f_\theta$ has Fourier support in a rectangle $\theta$ of dimensions $R \times R^{1/2} \times R^{1/2} \times \dots R^{1/2}$, and we let $u_\theta(x,t) = e^{it\sqrt{-\Delta}} f_\theta(x)$. Wolff (2000) suggested a possible approach to the local smoothing conjecture: the decoupling inequality

$$\|u\|_{L^p(\mathbb{R}^n \times [1,2])} \lessapprox \left( \sum_\theta \|u_\theta\|^p_{L^p(\mathbb{R}^n \times [1,2])} \right)^{1/p}, \tag{1}$$

where here $A \lessapprox B$ means that for every $\epsilon > 0$, there exists $C_\epsilon > 0$ such that $A \le C_\epsilon R^\epsilon B$ for all $R$. This is a subtle weakening of the inequality $A \lesssim B$.

The reasons to explore this inequality were presented in Lecture 7. We also stated the following result, due to Bourgain and Demeter (2015).

**Theorem 1.** *If $p \ge 2$ and $R \ge 1$, then*

$$\|u\|_{L^p} \lessapprox R^{\alpha(p)} \left( \sum_\theta \|u_\theta\|^p_{L^p} \right)^{1/p}, \tag{2}$$

*where* $\alpha(p) = \begin{cases} \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{p}\right), & \text{if } 2 \le p \le \frac{2(n+1)}{n-1} \\ (n-1)\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{p}, & \text{if } p \ge \frac{2(n+1)}{n-1}. \end{cases}$

Notice that $\frac{2(n+1)}{n-1}$ is the Tomas-Stein exponent. Today, we explore a bird's eye view of Bourgain and Demeter's approach to proving this.

## Dispensing with the Wave Equation

To prove Theorem 1, Bourgain and Demeter actually established a stronger inequality. Namely, they proved the following result.

**Theorem 2.** *If $p \ge 2$, $R \ge 1$, then*

$$\|u\|_{L^p} \lessapprox R^{d(p)} \left( \sum_\theta \|u_\theta\|^2_{L^p} \right)^{1/2}, \tag{3}$$

*where* $d(p) = \begin{cases} 0, & \text{if } 2 \le p \le \frac{2(n+1)}{n-1} \\ \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{p}, & \text{if } p \ge \frac{2(n+1)}{n-1}. \end{cases}$

First, we notice that this estimate holds at the endpoints. For $p = 2$, we have seen previously, using Plancherel's theorem, that $\|u\|_{L^2(\mathbb{R}^n \times [1,2])} = \|\hat{f}\|_{L^2}$, and in the same way $\|u_\theta\|_{L^2(\mathbb{R}^n \times [1,2])} = \|\hat{f}_\theta\|_{L^2}$. By construction, the functions $f_\theta$ have disjoint Fourier supports, so Pythagoras' theorem determines that $\|\hat{f}\|_{L^2} = \left( \sum_\theta \|\hat{f}_\theta\|^2_{L^2} \right)^{1/2}$, so the desired inequality is already known for $p = 2$.

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For $p = \infty$, we trivially have $\|u\|_{L^\infty} \leq \sum_\theta \|u_\theta\|_{L^\infty} \leq (\#\theta)^{1/2} \left(\sum_\theta \|u_\theta\|_{L^\infty}^2\right)^{1/2}$. Since $\#\theta \approx R^{\frac{n-1}{2}}$, this proves the required inequality for $p = \infty$.

Therefore, to prove Theorem 2, we only need to prove the $p = \frac{2(n+1)}{n-1}$ case.

To see why Theorem 2 implies Theorem 1, we use the generalised Hölder inequality: whenever $\frac{1}{q} + \frac{1}{p} = \frac{1}{r}$, we have $\|fg\|_{L^r} \leq \|f\|_{L^q}\|g\|_{L^p}$. In our case, set $r = 2$, and choose $q$ so that $\frac{1}{q} = \frac{1}{2} - \frac{1}{p}$. Since the number of rectangles $\theta$ is approximately $R^{\frac{n-1}{2}}$, we have

$$\begin{aligned} \left(\sum_\theta \|u_\theta\|_{L^p}^2\right)^{1/2} &\leq \left(\sum_\theta 1^q\right)^{1/q} \left(\sum_\theta \|u_\theta\|_{L^p}^p\right)^{1/p} \\ &\approx R^{\frac{n-1}{2}(\frac{1}{2}-\frac{1}{p})} \left(\sum_\theta \|u_\theta\|_{L^p}^p\right)^{1/p}. \end{aligned}$$

Since $d(p) + \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{p}\right) = \alpha(p)$, this proves Theorem 1.

It turns out that Theorem 2 can be proven in a much more general setting that is not explicitly connected to the wave equation at all. It turns out that Theorem 2 really depends on the Fourier supports of the functions $u_\theta$.

Recall that

$$u_\theta(x,t) = \int e^{2\pi i(x\cdot\xi + t|\xi|)} \hat{f}(\xi) \mathrm{d}\xi.$$

Since $x \cdot \xi + t|\xi| = (x,t)\cdot(\xi, |\xi|)$, this implies that the Fourier transform of $u$, as a function on $\mathbb{R}^{n+1}$, is supported on the subset $\{(\xi,|\xi|) : \xi \in \theta\}$ of the light cone in $\mathbb{R}^{n+1}$. However, since we are only considering $u_\theta(x,t)$ for $t \in [1,2]$, we are actually interested in the Fourier transform of the function $(x,t) \mapsto u_\theta(x,t)\mathbf{1}_{[1,2]}(t)$. In Lecture 2, we explored a heuristic that said $\mathbf{1}_{[1,2]}$ has Fourier support in $[-\frac{1}{2}, \frac{1}{2}]$. Since the Fourier transform interchanges products with convolutions, it follows that the function $u_\theta(x,t)\mathbf{1}_{[1,2]}(t)$ has Fourier support contained in a $\frac{1}{2}$-neighbourhood of the light cone. Therefore, we are led to reformulate Theorem 2 as follows.

**Theorem 3.** *Let $p \geq 2$, $R \geq 1$, and partition the annulus $\{|\xi| \sim R\}$ into rectangles $\theta$ as before. For each $\theta$, let $R_\theta$ be the 1-neighbourhood in $\mathbb{R}^{n+1}$ of the region $\{(\xi,|\xi|) \in \mathbb{R}^{n+1} : \xi \in \theta\}$ on the light cone, and let $u_\theta \in \mathscr{S}(\mathbb{R}^{n+1})$ be a function with* $\mathrm{supp}(\widehat{u_\theta}) \subset R_\theta$*. Then, with $d(p)$ defined as in Theorem 2,*

$$\|\sum_\theta u_\theta\|_{L^p} \lessapprox R^{d(p)} \left(\sum_\theta \|u_\theta\|_{L^p}^2\right)^{1/2}. \tag{4}$$

This is a decoupling estimate on the cone. Using Pramanik-Seeger iteration (2009), one can deduce this from a decoupling estimate for parabolas.

## $\ell^2$-PARABOLA DECOUPLING

For simplicity, we will consider only the case $n = 2$. The setup described in the forthcoming theorem is depicted in Figure 1.

**Theorem 4.** *Consider the arc $A = \{(\xi,\xi^2) : 0 \leq \xi \leq 1\}$ on the parabola in $\mathbb{R}^2$. Suppose $\Theta$ is a collection of rectangles with dimensions $R^{-1/2} \times R^{-1}$ that are roughly tangent to $A$ and cover an*

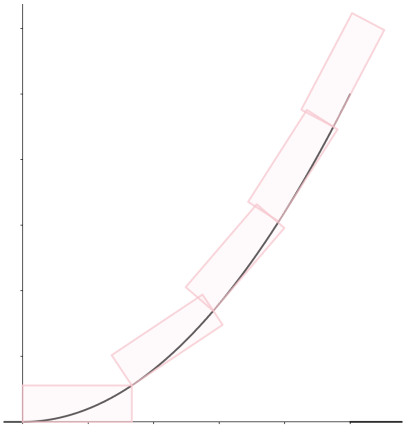

FIGURE 1. A collection of rectangles for parabola decoupling

*approximately* $R^{-1}$*-neighbourhood of* $A$*. For every rectangle* $\theta \in \Theta$*, let* $f_\theta \in \mathscr{S}(\mathbb{R}^2)$ *be a Schwartz function with* $\operatorname{supp}(\widehat{f_\theta}) \subset \theta$*. Then, for all* $R \geq 1$ *and* $p \geq 2$*, and with* $d(p)$ *defined as in Theorem 2,*

$$\|\sum_\theta f_\theta\|_{L^p} \lessapprox R^{d(p)} \left(\sum_\theta \|f_\theta\|_{L^p}^2\right)^{1/2}. \tag{5}$$

That is, we have bounded the left-hand side in terms of the $\ell^2$-norm of all the $\|f_\theta\|_{L^p}$.

First, let's consider how to use this result to deduce decoupling for cones. Suppose $f \in \mathscr{S}(\mathbb{R}^3)$ has Fourier support in a small neighbourhood $U$ of the light cone near the point $(0, t, t)$, $1 \leq t \leq 2$. (By rotational symmetry, this will apply to all points on the light cone.) Consider the change-of-coordinates $(\eta_1, \eta_2, \eta_3) = T(\xi_1, \xi_2, \xi_3) := (\xi_1, \xi_2, \xi_3 - \xi_2)$, so that the light cone is defined by the equation $\eta_3 = \sqrt{\eta_1^2 + \eta_2^2} - \eta_2$, as depicted in Figure 2. Then $f \circ (T^{-1})^t$ has Fourier support in $T(U)$, which is a neighbourhood of the rotated cone near $(0, t, 0)$. Near this point, a Taylor expansion shows that the equation defining the light cone is roughly the equation for a parabola in the $(\eta_1, \eta_3)$-plane. Now, we can apply the $\ell^2$-decoupling estimate for the parabola, along with Fubini's theorem, to deduce the desired estimate for the cone. (For details, see these notes.)

We will not attempt to complete Bourgain and Demeter's proof by proving Theorem 4. However, we present a calculation that suggests it should be achievable. The key idea is *induction on scales*. Since the number of rectangles in $\Theta$ is approximately $R^{1/2}$, we obtain the trivial bound $\|\sum_\theta f_\theta\|_{L^p} \leq \sum_\theta \|f_\theta\|_{L^p} \leq R^{1/4} \left(\sum_\theta \|f_\theta\|_{L^p}^2\right)^{1/2}$ by the Cauchy-Schwarz inequality. Therefore, (5) holds trivially if we assume, say, $R \leq N$.

Now, suppose $R \leq N^3$, and set $S = \frac{R}{N^2}$. Then, for any collection of rectangles $\Theta$ as in Theorem 4, we can group sets of $N$ consecutive rectangles together and contain each group inside a rectangle $\tau$ with dimensions $S^{-\frac{1}{2}} \times S^{-1}$. Since $S \leq N$, we have

$$\left\|\sum_\theta f_\theta\right\|_{L^p(\mathbb{R}^2)} = \|\sum_\tau \left(\sum_{\theta \subset \tau} f_\theta\right)\|_{L^p(\mathbb{R}^2)} \lessapprox S^{d(p)} \left(\sum_\tau \left\|\sum_{\theta\subset\tau} f_\theta\right\|_{L^p}^2\right)^{1/2}.$$

By adjusting the induction hypothesis, we can also deduce that

$$\left\|\sum_{\theta\subset\tau} f_\theta\right\|_{L^p(\mathbb{R}^2)} \lessapprox \left(\frac{R}{S}\right)^{d(p)} \left(\sum_{\theta\subset\tau} \|f_\theta\|_{L^p}^2\right)^{1/2}.$$

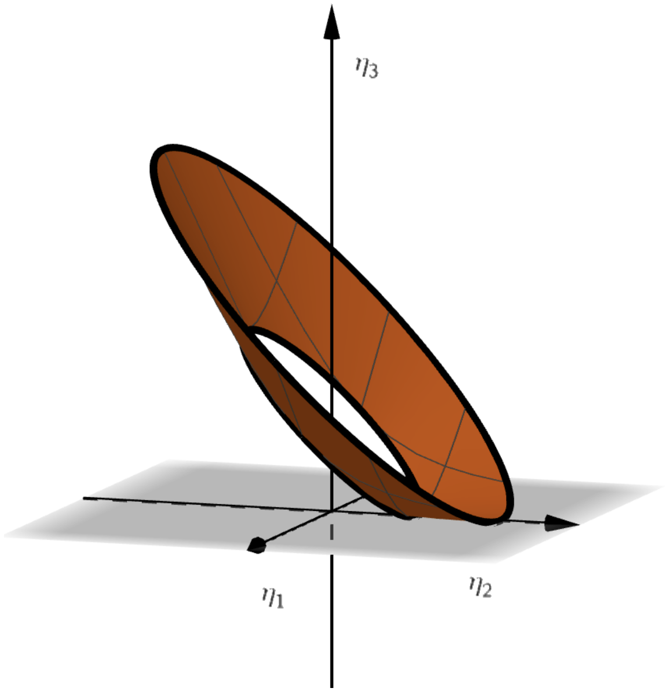

FIGURE 2. The truncated light cone in coordinates $(\eta_1, \eta_2, \eta_3)$

Altogether, this shows that $\|\sum_\theta f_\theta\|_{L^p(\mathbb{R}^2)} \lessapprox R^{d(p)} \left(\sum_\theta \|f_\theta\|_{L^p}^2\right)^{1/2}$.

Unfortunately, such an induction argument, at least as stated here, cannot work, because the implicit constants will accumulate as we repeatedly apply the induction step. However, it might give us some hope that a similar argument could be accessible! As a comparison point, if we were to attempt such an argument for reversed square function estimates (see Lecture 7), we would run into a dead-end immediately. We try

$$\left\|\sum_\theta f_\theta\right\|_{L^p} = \left\|\sum_\tau \left(\sum_{\theta\subset\tau} f_\theta\right)\right\|_{L^p} \lessapprox \left\|\left(\sum_\tau \left|\sum_{\theta\subset\tau} f_\theta\right|^2\right)^{1/2}\right\|_{L^p},$$

but then it is not clear how to induct on this.

## CONNECTION TO DISCRETE RESTRICTION ESTIMATES

Let us return to the general setting of $\mathbb{R}^n$. The Tomas-Stein exponent appears in $d(p)$ for good reason: decoupling estimates for paraboloids are related to a discrete analogue to the Tomas-Stein theorem for paraboloids. Letting $\mathrm{d}\sigma$ be the surface measure of the paraboloid $P$ in $\mathbb{R}^n$, for any $f \in L^2(P)$ and $x \in \mathbb{R}^n$, we set $(f\mathrm{d}\sigma)^\vee(x) = \int_P f(\xi)e^{ix\cdot\xi}\mathrm{d}\sigma(\xi)$. The Tomas-Stein estimate says that the mapping $f \mapsto (f\mathrm{d}\sigma)^\vee$ is bounded from $L^2(P)$ to $L^{\frac{2(n+1)}{n-1}}(\mathbb{R}^n)$.

Consider the discrete analogue of this setup. Let $P_\mathbb{Z}$ be the "discrete paraboloid",

$$P_d := \left\{(k_1, \dots, k_n) \in \mathbb{Z}^n : k_n = \sum_{i=1}^{n-1} k_i^2\right\}.$$

The correct image of the Fourier transform on $\mathbb{Z}$ is the interval $[0,1]$. Therefore, for any function $f \in \ell^2(P_d)$ and $x \in [0,1]^n$, we define $f^\vee(x) = \sum_{k\in P_d} f(k)e^{2\pi i k\cdot x}$. Hence, we ask whether the mapping $f \mapsto f^\vee$ is a bounded map from $\ell^2(P_d)$ to $L^{\frac{2(n+1)}{n-1}}([0,1]^n)$. The answer is no! However, we do have the following result.

If the function $f$ on $P_d$ is supported in the Euclidean ball of radius $N$, then

$$\|f^\vee\|_{L^{\frac{2(n+1)}{n-1}}([0,1]^n)} \lesssim_\epsilon N^\epsilon \|f\|_{\ell^2}.$$

Zane Li and Wooley have used ideas related to this result to give a simpler proof of $\ell^2$-decoupling for the parabola in $\mathbb{R}^2$.