LECTURE 3

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The Laplacian

$$\Delta = \sum_i \frac{\partial^2}{\partial x_i^2}$$

satisfies the equation (at least formally)

$$-\Delta f(x) = \int_{\mathbb{R}^n} (2\pi|\xi|)^2 \hat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi$$

due to integration by parts. The factor $(2\pi|\xi|)^2$ is sometimes called the "symbol" of the Laplacian, which in some other contexts has a different normalisation due to alternate definitions of the Fourier transform.

We formally define

$$m(\sqrt{-\Delta}) = \int_{\mathbb{R}^n} m(2\pi|\xi|)\hat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi$$

for functions m. In particular,

$$e^{\pm it\sqrt{-\Delta}} = \int_{\mathbb{R}^n} e^{\pm 2\pi i t|\xi|}\hat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi$$

is of interest, as these are the solution operators to the wave equation

$$\begin{cases} \partial_t^2 u = \Delta u \\ u(x,0) = f(x) \\ \partial_t u(x,0) = 0 \end{cases}$$

in the sense that

$$u(x,t) = \frac{1}{2} e^{it\sqrt{-\Delta}} f(x) + \frac{1}{2} e^{-it\sqrt{-\Delta}} f(x)$$

These two parts behave very similarly (except that they propagate in opposite directions), so we restrict our attention to just the positive one.

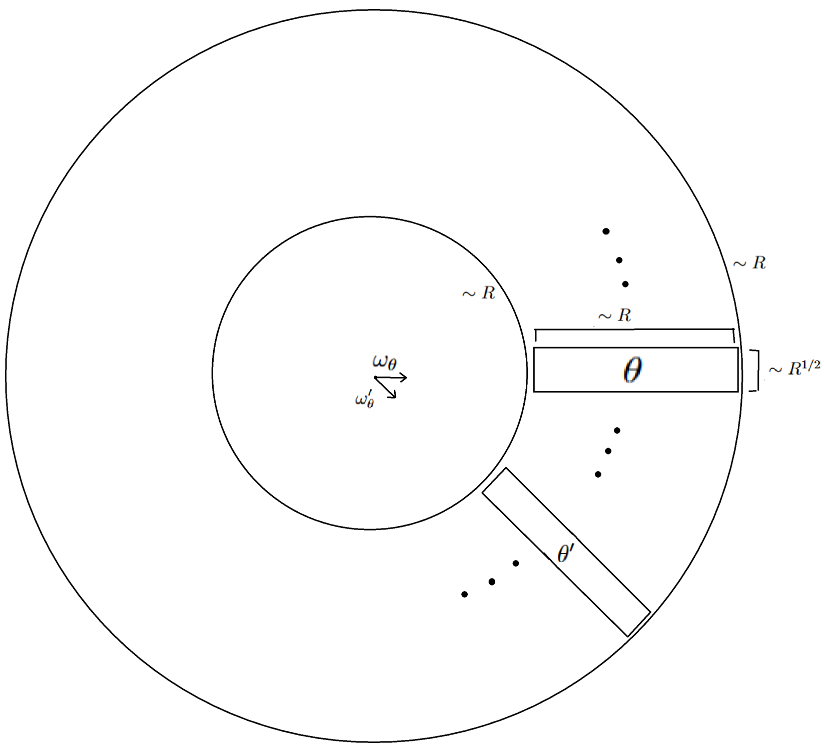
To understand the concentration of u at $t = 1$ (or any time near one), we want to compare the L^p norm of $u(x,1)$ to that of $u(x,0) = f(x)$ for very large p; ideally for $p = \infty$ as this will be the closest to what we literally are interested in. But let us start smaller: At $p = 2$,

$$||u(x,1)||_{L^2(\mathbb{R}^n)} = \left|\left|\int_{\mathbb{R}^n} e^{2\pi i|\xi|}\hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi\right|\right|_{L^2(\mathbb{R}^n)}$$

$$= ||e^{2\pi i|\xi|}\hat{f}(\xi)||_{L^2(\mathbb{R}^n)} = ||\hat{f}||_{L^2(\mathbb{R}^n)} = ||f||_{L^2(\mathbb{R}^n)}$$

by two applications of Plancherel's theorem. So at least for $p = 2$, the concentration is preserved as the solution evolves. This is sometimes loosely referred to as conservation of energy.

Now let us look at the other extreme; $p = \infty$. Suppose $\text{supp}\hat{f} \subset \{|\xi| \approx R\}$, that is, a large annulus. We cover this annulus by rectangular boxes θ with radial dimension approximately R, and angular dimensions approximately $R^{1/2}$:



Then split up $\hat{f} = \sum_\theta \hat{f}_\theta$, where $\hat{f}_\theta$ is supported on θ. These $\hat{f}_\theta$ are called "wave packets". Denote ω_θ to be unit vectors pointing towards the boxes θ. If ϕ_θ is a smooth function that is identically 1 on θ and supported in 2θ, then

$$e^{it\sqrt{-\Delta}}f_\theta(x) = \int_{\mathbb{R}^n}\widehat{f_\theta}(\xi)\phi_\theta(\xi)e^{2\pi i(t|\xi|+x\cdot\xi}d\xi = \int_{\mathbb{R}^n} f_\theta(x-y)\int_{\mathbb{R}^n}\phi_\theta(\xi)e^{2\pi i(t|\xi|+y\cdot\xi)}d\xi dy.$$

We Taylor expand

$$\begin{aligned} |\xi| &= |R\omega_\theta| + \omega_\theta \cdot (\xi - R\omega_\theta) + O(|\xi - R\omega_\theta|^2) \\ &= \omega_\theta \cdot \xi + O(|\xi - R\omega_\theta|^2) \end{aligned}$$

and note that the quadratic error actually satisfies

$$|\partial_\xi^\alpha (\omega_\theta \cdot \partial_\xi)^M (|\xi| - \omega_\theta \cdot \xi)| \lesssim R^{-|\alpha|/2-M}$$

for every α and M if $\xi \in 2\theta$. In particular,

$$|\partial_\xi^\alpha (\omega_\theta \cdot \partial_\xi)^M e^{2\pi it(|\xi|-\omega_\theta\cdot\xi)}| \lesssim R^{-|\alpha|/2-M}$$

for every α and M, valid for $\xi \in 2\theta$ and $|t| \lesssim 1$, so if

$$K_{\theta,t}(y) := \int_{\mathbb{R}^n} \phi_\theta(\xi) e^{2\pi it(|\xi|-\omega_\theta\cdot\xi)} e^{2\pi i+y\cdot\xi} d\xi$$

then we have

$$e^{it\sqrt{-\Delta}} f_\theta(x) = f_\theta * K_{\theta,t}(x + t\omega_\theta)$$

where $K_{\theta,t}$ satisfies, for $|t| \lesssim 1$, an estimate

$$|K_{\theta,t}(y)| \lesssim_N \frac{1}{|\theta^*|}(1 + R|y\cdot\omega_\theta| + |y|R^{1/2})^{-N}$$

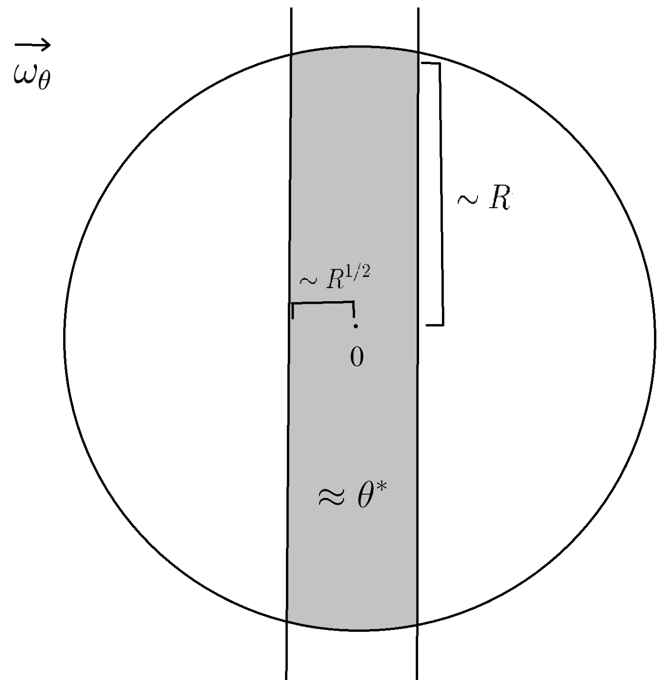
for every $N > 0$. This says if either

$$|y \cdot \omega_\theta| \geq \frac{1}{R}$$

or

$$|y| \geq \frac{1}{R^{1/2}}$$

then $K_{\theta,t}(y)$ will become very small. The complement of this area is the intersection of a strip about the origin orthogonal to ω_θ of radius $1/R$ and a ball of radius $1/R^{1/2}$. That is, almost exactly the dual box θ^*.



Since $K_{\theta,t}$ is bounded above by $1/|\theta^*|$ everywhere, it is in some sense approximately equal to the normalised indicator function

$$K_{\theta,t}(y) \approx \frac{1}{|\theta^*|} 1_{\theta^*}(y)$$

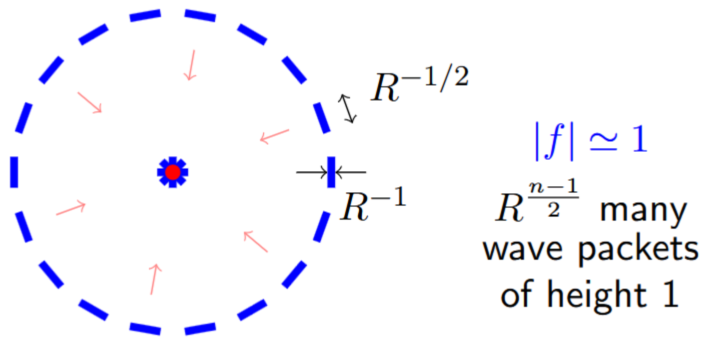
and thus the convolution is approximately

$$e^{it\sqrt{-\Delta}} f_\theta(x) \approx f_\theta(x + t\omega_\theta)$$

as by the heuristics of the previous lecture, f_θ is roughly constant on translates of its supporting box.

So although the solution operator of the wave equation is in general quite complicated, on a single wave packet it approximately looks like translation toward (or away from) the origin.

Now let us be even more particular about the f we are considering. Suppose that $|f|$ is approximately the indicator function of a spherical shell of radius 1 and thickness $1/R$ (since $\hat{f}_\theta$ are approximately indicator functions of the boxes θ, we can arrange this by modulating by $e^{-2\pi i \omega_\theta \cdot \xi}$, which will appropriately translate the packets in position space).



Then at time 1, all the wave packets will evolve to the origin in position space. The angular cross-sectional volume of each packet in position space is $(R^{-1/2})^{n-1}$, so there will be approximately $CR^{(n-1)/2}$ of them, where C is the surface area of a sphere of radius 1. Each packet has height one, so when they all overlap and constructively interfere at $t = 1$, we will have

$$||u(x,1)||_{L^\infty(\mathbb{R}^n)} \approx R^{\frac{n-1}{2}}$$

But the time 0 norm is trivially 1. So since we can vary R, there cannot possibly be any way to control the L^∞ concentration at time 1 in terms of the concentration at time 0.

For $2 < p < \infty$, we get a similar estimate:

$$||u(x,0)||_{L^p(\mathbb{R}^n)} = ||f||_{L^p(\mathbb{R}^n)} = (CR^{-1})^{1/p} = C'R^{-1/p}$$

and at time 1, we restrict our attention to a ball of radius $1/R$, since that is the radial width of each packet:

$$\begin{aligned} ||u(x,1)||_{L^p(\mathbb{R}^n)} &\geq \left(\int_{B_0(1/R)} |u(x,1)|^p dx\right)^{1/p} \\ &\gtrsim ||u(x,1)||_{L^\infty(\mathbb{R}^n)}(\text{vol}(B_0(1/R)))^{1/p} \simeq R^{\frac{n-1}{2}}R^{-\frac{n}{p}} \end{aligned}$$

So the relative concentration is

$$\frac{||u(x,1)||_{L^p(\mathbb{R}^n)}}{||u(x,0)||_{L^p(\mathbb{R}^n)}} \gtrsim R^{(n-1)(\frac{1}{2}-\frac{1}{p})}$$

The reason we have calculated the exponent of the R explicitly instead of immediately giving up upon seeing it is at least one is that we can now use the Littlewood-Paley decomposition to say that

$$||u(x,1)||_{L^p(\mathbb{R}^n)} \gtrsim ||f||_{W^{(n-1)(\frac{1}{2}-\frac{1}{p}),p}(\mathbb{R}^n)}$$

This then motivates the exponent in the following

Theorem (Peral, Miyachi): If $s = (n-1)(\frac{1}{2} - \frac{1}{p})$, then for any $f \in W^{s,p}(\mathbb{R}^n)$, the solution of the wave equation $u(x,t) = e^{it\sqrt{-\Delta}} f(x)$ satisfies

$$||u(x,1)||_{L^p(\mathbb{R}^n)} \lesssim ||f||_{W^{s,p}(\mathbb{R}^n)}$$

That is, the example f we were working with was particularly pathological, in that it makes the bound in this theorem sharp.

So this is nice. We have found the best possible description of fixed time concentration of solutions to the wave equation. What about space-time estimates?

If our pathological example is still the worst, we expect the bound to be slightly better than the fixed time case: Since the wave packets have radial length $1/R$, they will overlap for about $1/R$ time. So,

$$\begin{aligned} ||u(x,t)||_{L^p(\mathbb{R}^n_x \times [1,2]_t)} &= \left(\int_1^2 ||u(x,t)||^p_{L^p(\mathbb{R}^n_x)} dt \right)^{1/p} \\ &\gtrsim (1/R)^{1/p} ||u(x,1)||_{L^p \mathbb{R}^n_x} \end{aligned}$$

for p large.

This leads us to the (naive, first statement) of the local smoothing conjecture:

Local smoothing conjecture Version 1: If $u(x,t) = e^{it\sqrt{-\Delta}} f(x)$ with supp$\hat{f} \subset \{|\xi| \approx R\}$, then

$$||u(x,t)||_{L^p(\mathbb{R}^n_x \times [1,2]_t)} \lesssim R^{(n-1)(\frac{1}{2} - \frac{1}{p}) - \frac{1}{p}} ||f||_{L^p(\mathbb{R}^n)}$$

for $p \geq \frac{2n}{n-1}$.

We will see very soon that this is false as stated, and needs to be modified. To get some more intuition about the conjecture, let's consider the case $n = 2$. Our example with the spherical shell was extreme in the amount of overlap, but very timid in the overlapping time. So let us consider two other examples with longer time overlap:

(1) Arc of the shell:

Here we consider our previous example, but with the support of f restricted to a small subset of the shell (say $sR^{1/2}$ many wave packets θ, which substend an angle s on an arc of angle s).

Since the angle between the wave packets is now quite small, they will overlap both for a longer time (in fact, for a time interval of length $(sR^{1/2})^{-2}$), and in more area (rectangle of dimension $\sim (R^{-1}/s) \times R^{-1}$ instead of $\sim R^{-1} \times R^{-1}$).

But there are also significantly less packets to overlap, so the height is much smaller (only $sR^{1/2}$). It turns out these differences exactly cancel, and in this case for $p = 4$ we have $\|u(x,t)\|_{L^p(\mathbb{R}^2_x \times [1,2]_t)} \gtrsim (sR^{1/2})^{-2/p}(R^{-1}/s \times R^{-1})^{1/p}(sR^{1/2}) \simeq s^{1-3/p}R^{1/2-3/p}$ whereas $\|f\|_{L^p(\mathbb{R}^2)} = (sR^{-1})^{1/p}$ so

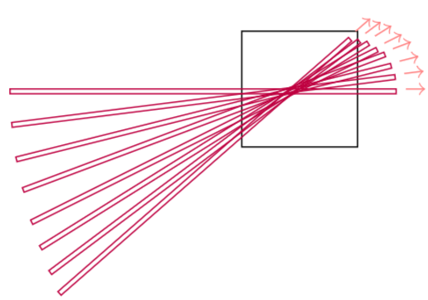
$$\|u(x,t)\|_{L^4(\mathbb{R}^2_x \times [1,2]_t)} / \|f\|_{L^4(\mathbb{R}^2)} \geq 1$$

and this is consistent with version 1 of the conjecture.

(2) Wave trains:

Here $\hat{f}$ is supported on a cylinder of length 1 which is split into R packets of length $1/R$ and width $1/R^{1/2}$. The packets once again approximately evolve towards or away from the origin, so the entire train will behave this way.

If we take many of these trains at different angles, they will overlap for much longer times, or even **all** times $t \in [1,2]$:



So it would make sense for this example to be pathological in the space-time case, and in fact it is. This is inconsistent with our naive version of local smoothing conjecture.

So, we must modify the exponent in order to expect the conjecture may be true. We will also put a small constraint on p so that everything makes sense:

Local smoothing conjecture (Good Version): If $u(x,t) = e^{it\sqrt{-\Delta}}f(x)$ with supp$\hat{f} \subset \{|\xi| \approx R\}$ and $p \geq \frac{2n}{n-1}$, then for all $\epsilon > 0$,

$$||u(x,t)||_{L^p(\mathbb{R}^n_x \times [1,2]_t)} \lesssim_\epsilon R^{(n-1)(\frac{1}{2}-\frac{1}{p})-\frac{1}{p}+\epsilon}||f||_{L^p(\mathbb{R}^n)}$$

The $n = 2$ case of this was proven in 2020 by Guth, Wang and Zhang.

Since the wave trains example is the worst known, local smoothing conjecture is related to the "Kakeya Problem", which is about how tubes overlap in $\mathbb{R}^n$. The nature of this relation is that local smoothing conjecture implies the "Kakeya Conjecture".

Relation to Bochner-Riesz:

By some non-trivial complex analysis, it can be shown that

$$\int_{\mathbb{R}} \frac{e^{-it\tau} - 1}{t^{1+\delta}} dt = c_\delta \tau_+^\delta$$

for $0 < \delta < 1$, where $c_\delta \neq 0$ is a constant, and

$$\tau_+^\delta = \begin{cases} \tau^\delta, & \tau \geq 0 \\ 0, & \text{else} \end{cases}$$

This is relevant as the Bochner-Riesz multiplier can be written as

$$(1 - |\xi|^2)_+^\delta = ((1 + |\xi|)(1 - |\xi|))_+^\delta \simeq (1 - |\xi|)_+^\delta$$

on the ball of radius 1. Thus, (working heuristically from this point forwards), we have

$$\begin{aligned} S^\delta f &\simeq \int_{\mathbb{R}^n} \int_{\mathbb{R}} \frac{e^{-it(1-|\xi|)} - 1}{t^{1+\delta}} e^{ix\cdot\xi} \hat{f}(\xi) dt d\xi \\ &= \int_{\mathbb{R}} \frac{e^{-it}u(x,t) - f(x)}{t^{1+\delta}} dt \end{aligned}$$

where

$$u(x,t) = e^{it\sqrt{-\Delta}}f(x) = \int_{\mathbb{R}^n} e^{2\pi i|\xi|} \hat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi$$

is a solution of the wave equation.

The point here (which will be expanded on in the next lecture), is that the L^p norm of the Bochner-Riesz operator is thus controlled by the concentration of the wave equation, in some sense. It turns out that in fact, the Local Smoothing Conjecture implies the Bochner-Riesz conjecture as well. So local smoothing conjecture is quite a strong conjecture as it implies several other important results in harmonic analysis.