LECTURE 6

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In this lecture we explore the ideas behind the Fourier restriction conjecture.

1. Motivation, and a baby case for restriction on the sphere

Let $f \in L^p(\mathbb{R}^n)$, $p \in [1, 2]$. If $p = 1$, then by the dominated convergence theorem, the Fourier transform converges to a continuous function on $\mathbb{R}^n$. On the other hand, for $p \in (1, 2]$, this need not happen and so the Fourier transform is only defined up to almost everywhere equivalence. In particular, we cannot make sense of $\hat{f}$ restricted to a hyperplane.

What Stein realised however, is that for appropriate p, we **can** restrict to a curved hypersurface (explicitly, a surface with nonvanishing Gaussian curvature. In the below we will just consider a sphere):

We define the operator $Rf = \hat{f}|_{\mathbb{S}^{n-1}}$ for Schwartz functions (as these are in particular L^1). Fix $q > 1$. If we can find some $r \geq 1$ such that for every $f \in \mathcal{S}(\mathbb{R}^n)$,

$$||Rf||_{L^r(\mathbb{S}^{n-1})} \lesssim ||f||_{L^q(\mathbb{R}^n)}$$

then for a general $f \in L^q(\mathbb{R}^n)$ we could pick sequences $(f_j)_j \subset \mathcal{S}(\mathbb{R}^n)$ with $f_j \to f$ in $L^q(\mathbb{R}^n)$. The above bound would then force $(Rf_j)_j \subset L^r(\mathbb{S}^{n-1})$ to be Cauchy, and thus converge to some $Rf \in L^r(\mathbb{S}^{n-1})$. It can be verified that this is independent of the choice of Cauchy sequence, and thus it defines a bounded linear operator

$$R : L^q(\mathbb{R}^n) \to L^r(\mathbb{S}^{n-1})$$

But when do we have such an r?

The easiest exponent to work with is always 2, so let's start there: If

$$R : L^q(\mathbb{R}^n) \to L^2(\mathbb{S}^{n-1})$$

then

$$R^* : L^2(\mathbb{S}^{n-1}) \to L^{q'}(\mathbb{R}^n)$$

and so

$$R^*R : L^q(\mathbb{R}^n) \to L^{q'}(\mathbb{R}^n)$$

In fact, by examining the inner product on $L^2(\mathbb{R}^n)$, we see this is if and only if. For our particular R, we thus want to understand R^*R, so we should at least try to understand R^*:

$$\begin{aligned}\langle R^*g, f\rangle = \langle g, Rf\rangle &= \int_{\mathbb{S}^{n-1}} g(\xi)\overline{\hat{f}(\xi)}d\sigma(\xi) \\ &= \int_{\mathbb{S}^{n-1}} g(\xi)\overline{\int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} f(x)dx}d\sigma(\xi) \\ &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{S}^{n-1}} e^{2\pi i x\cdot\xi} g(\xi)d\sigma(\xi)\right)\overline{f(x)}dx \\ &= \langle (gd\sigma)^\vee, f\rangle\end{aligned}$$

as long as $f \in \mathcal{S}(\mathbb{R}^n)$, $g \in L^1(\mathbb{R}^n)$. Here, as in previous lectures, $d\sigma$ is the normalised surface measure of the sphere. We denote this as $R^*g = \mathcal{E}g = (gd\sigma)^\vee$, where of course $\mathcal{E}$ stands for Extension, dually to R standing for Restriction.

We then have

$$R^*Rf = (\hat{f}d\sigma)^\vee = f * (d\sigma^\vee)$$

By symmetry, $d\sigma^\vee = d\sigma^\wedge$ which from previous lectures behaves as

$$|d\sigma^\wedge| \leq (1+|x|)^{-\frac{n-1}{2}}$$

Since

$$\left((1+|x|)^{-\frac{n-1}{2}}\right)^{\frac{2n}{n-1}} = (1+|x|)^{-n}$$

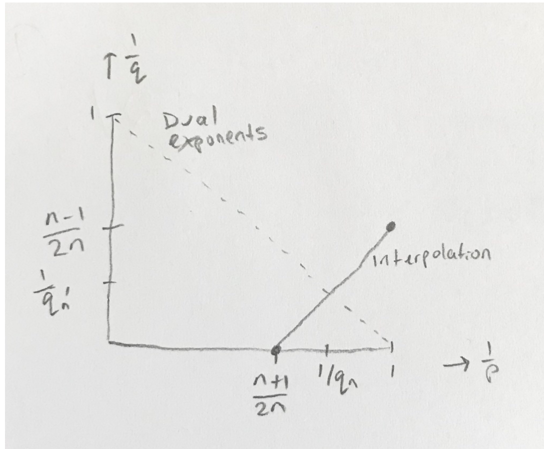
we have that $d\sigma^\vee$ is *almost* in $L^{\frac{2n}{n-1}}(\mathbb{R}^n)$. We were not being very precise already so let us pretend this is true. Convolution with a fixed function in $L^{\frac{2n}{n-1}}(\mathbb{R}^n)$ will map

$$L^1(\mathbb{R}^n) \to L^{\frac{2n}{n-1}}(\mathbb{R}^n)$$

and dually,

$$L^{\frac{2n}{n+1}}(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$$

This then gives us two points in the p, q square, between which we can interpolate:



As can be seen in the above diagram, there is a pair of dual exponents along this interpolation line, and thus some q_n for which R^*R maps $L^{q_n}(\mathbb{R}^n) \to L^{q_n'}(\mathbb{R}^n)$, which means we have found a case where $r = 2$ for an appropriate q (as in the setup of this lecture).

2. Tomas-Stein restriction theorem

A more careful treatment of the above ideas leads us to the

Tomas-Stein Theorem: The extension operator maps $L^2(\mathbb{S}^{n-1}) \to L^{\frac{2(n+1)}{n-1}}(\mathbb{R}^n)$

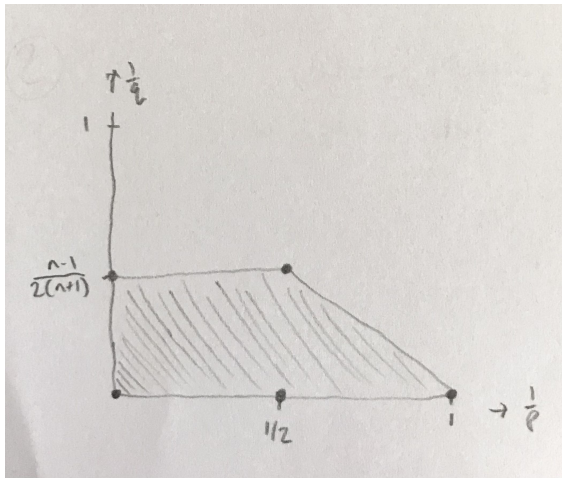
This doesn't seem like much at first, but via interpolation we can actually say a fair bit about $L^q \to L^p$ boundedness of $\mathcal{E}$:
$\mathbb{S}^{n-1}$ is compact, so

$$L^\infty(\mathbb{S}^{n-1}) \hookrightarrow L^2(\mathbb{S}^{n-1}) \hookrightarrow L^1(\mathbb{S}^{n-1}) \xrightarrow{\mathcal{E}} L^\infty(\mathbb{R}^n)$$

gives us 3 points on the $1/p, 1/q$ plot (since $||\mathcal{E}g||_{L^\infty(\mathbb{R}^n)} \leq \int |g| d\sigma = ||g||_{L^1(\mathbb{S}^{n-1})}$), and

$$L^\infty(\mathbb{S}^{n-1}) \hookrightarrow L^2(\mathbb{S}^{n-1}) \xrightarrow{\mathcal{E}} L^{\frac{2(n+1)}{n-1}}(\mathbb{R}^n)$$

gives us 2 more. Interpolating between all of these points gives us the area



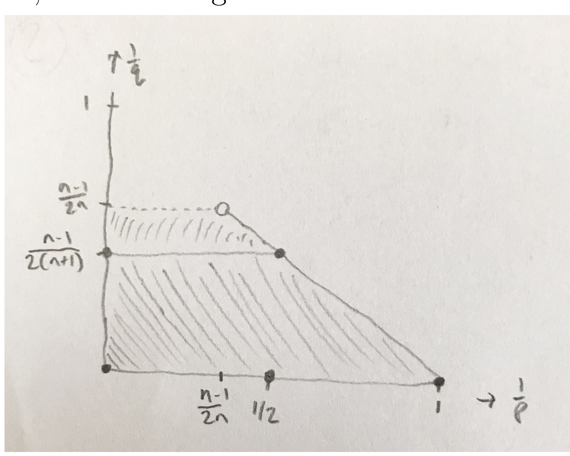
which is quite substantial.

There are analogs for other surfaces (such as paraboloids or cones in place of spheres), and they lead to Strichartz estimates in PDE.

3. The restriction conjecture

Now we can finally state the

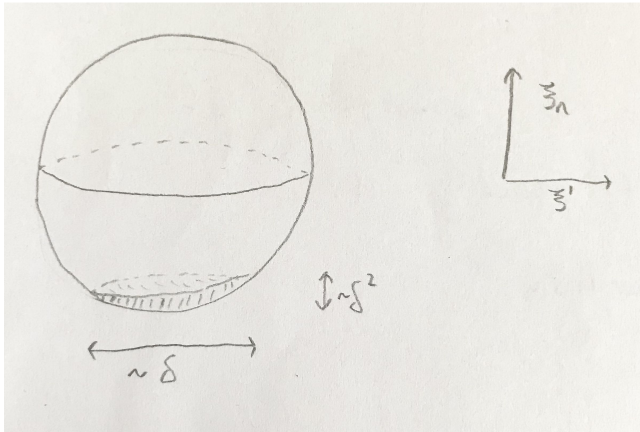
Fourier restriction conjecture (for spheres): The extension operator $\mathcal{E}$ is a bounded map from $L^q(\mathbb{S}^{n-1})$ to $L^p(\mathbb{R}^n)$ if $p > \frac{2n}{n-1}$ and $q' \leq \frac{n-1}{n+1}p$. That is, the following additional area:



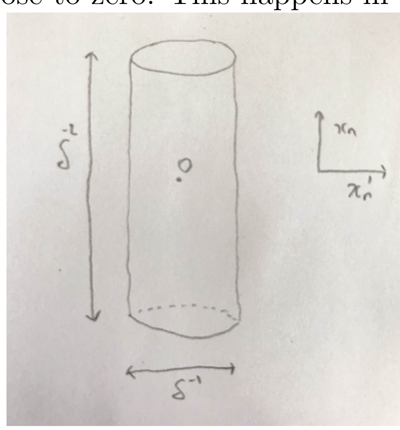
Let's now see that this conjecture is optimal via two examples:

(1) If $g \equiv 1$, g is in any $L^p(\mathbb{S}^{n-1})$ since $\mathbb{S}^{n-1}$ is compact. But $\mathcal{E}g(x) = d\sigma^\vee(x) \approx (1+|x|)^{-\frac{n-1}{2}} \notin L^p(\mathbb{R}^n)$ if $p \leq \frac{2n}{n-1}$. So the conjecture is optimal in p

(2) Now let g be the indicator function of a small spherical cap:



The volume of this cap is about δ^{n-1}, so $||g||_{L^q(\mathbb{S}^{n-1})} \approx \delta^{\frac{n-1}{q}}$. The integrand $e^{2\pi i x\cdot\xi}$ will be approximately constant when the phase $2\pi i x \cdot \xi$ is close to zero. This happens in the region



since inside this cylinder,

$$\begin{aligned} |x\cdot\xi| &\le |x_n\cdot\xi_n| + |x'\cdot\xi'| \\ &\le \delta^{-2}\delta^2 + \delta\delta^{-1} \approx 1 \end{aligned}$$

Thus,

$$|\mathcal{E}g(x)| = \left| \int_{\xi\in\text{cap}} e^{2\pi i x\cdot\xi} \right| \gtrsim \text{vol(cap)} = \delta^{n-1}$$

for x inside this cylinder, so when we take the norm,

$$\begin{aligned} ||\mathcal{E}g(x)||_{L^p(\mathbb{R}^n)} &\gtrsim \delta^{n-1}\left(\text{vol(Tube)}\right)^{1/p} \\ &\approx \delta^{n-1}\delta^{-\frac{n+1}{p}} \end{aligned}$$

This holds for any δ, and so in particular when $\delta \to 0$. For $\mathcal{E}$ to be a bounded map $L^q(\mathbb{S}^{n-1}) \to L^p(\mathbb{R}^n)$, we must then have a constant M with

$$\delta^{n-1}\delta^{-\frac{n+1}{p}} \leq M\delta^{\frac{n-1}{q}} \quad \forall \delta > 0$$

That is,

$$n - 1 - \frac{n+1}{p} \geq \frac{n-1}{q}$$

which eventually gives

$$\frac{n+1}{p} \leq \frac{n-1}{q'}$$

as a necessary condition on p and q for the conjecture to hold. So the conjectured region is sharp in q also.

(Note: if we had thickened the spherical cap to a small cylinder, these arguments are essentially the same as in earlier lectures when we analysed wave packets)

In $n = 2$, the restiction conjecture is completely solved due to Fefferman and Zygmund. In $n = 3$, there are a series of partial results which broadly make use of the following techniques:

(1) Splitting up

$$\mathcal{E}g = \sum_j \mathcal{E}g_j$$

allows us to express $||\mathcal{E}g||^p_{L^p(\mathbb{R}^n)}$ in terms of integrals

$$\int \left(|\mathcal{E}g_j(x)|^{1/2}|\mathcal{E}g_k(x)|^{1/2}\right)^{1/p} dx$$

For some appropriate choice of partition, $\mathcal{E}g_j$ and $\mathcal{E}g_k$ can be made in some sense transverse for $j \neq k$. This technique is called "bilinearisation", and was pioneered by Tao, Vargas and Vega in 1998 (with roots going back to breakthrough work of Klainerman and Machedon on bilinear estimates for wave equations). It has since become a standard tool in restriction theory. See also the multilinearisation method of Bourgain and Guth (2011), multilinear restriction theorem of Bennett, Carbery and Tao (2005), and subsequent contributions of Guth and others.

(2) Another major technique is to study the incidence geometry of tubes pointing in different directions. This was first used by Bourgain in 1991, and is helpful in particular in going below the Tomas-Stein exponent $p = 2(n + 1)/(n - 1)$.

(3) A 'divide and conquer' technique due to Guth is as follows: If we cut $\mathbb{R}^3$ into open subsets U_i via varieties (zero sets of real polynomials) such that

$$\int_{\mathbb{R}^3} |\mathcal{E}g(x)|^{3.25} dx = \sum_i \int_{U_i} |\mathcal{E}g(x)|^{3.25} dx$$

such that each individual integral is approximately the same as i varies, some algebraic geometry/topology techniques including the Borsuk-Ulam theorem can be used to get a bound for $p = 3.25$ in dimension $n = 3$ (higher dimensional improvements are also possible along this line). This idea has its roots in Dvir's famous solution of the finite field Kakeya conjecture. Since we are cutting $\mathbb{R}^3$ into open sets using zero sets of polynomials, this method is now known as the polynomial partitioning method. There has been many important contributions along these lines, both for restriction and Kakeya problems, notably by Jonathan Hickman, Keith Rogers and Ruixiang Zhang.

(4) Very recently, Shukun Wu and Hong Wang (arXiv:2411.08871) found a way of reducing the restriction conjecture entirely to an incidence estimate: very roughly speaking, they consider the amount of space one can occupy in $\mathbb{R}^3$ if one shades an s dimensional subset on a r dimensional family of unit line segments without concentrating the shading on one end of any line segment. This builds upon earlier work of Kevin Ren and Hong Wang on the Furstenberg conjecture in the plane, which in turn relies on substantial effort of many other authors in geometric measure theory.

4. Bochner-Riesz implies restriction

We now explore how the restriction conjecture relates to Bochner-Riesz:

Theorem (Tao): Suppose for every $\alpha > 0$,

$$||S^\alpha f||_{L^p(\mathbb{R}^n)} \lesssim ||f||_{L^p(\mathbb{R}^n)}$$

whenever

$$\frac{2n}{n+1+2\alpha} < p < \frac{2n}{n-1-2\alpha}$$

Then the restriction operator $\mathcal{E}$ satisfies

$$||\mathcal{E}g||_{L^p(\mathbb{R}^n)} \lesssim ||g||_{L^p(\mathbb{S}^{n-1})}$$

for every $p > \frac{2n}{n-1}$

So if Bochner-Riesz holds, we obtain these additional points in the $p-q$ boundedness chart of the restriction operator, which via interpolation will imply the entire conjecture.

In fact, using a more advanced argument than Riesz-Thorin interpolation, an $L^\infty \to L^p$ bound would be sufficient to prove Fourier restriction. But we will not discuss this.

We now sketch the proof of this theorem:

$$S^\alpha f \approx \int_{\mathbb{R}^n} f(y) \frac{e^{\pm i|x-y|}dy}{(x-y)^{\frac{n+1}{2}+\alpha}}$$

from previous lectures. Decomposing (via a partition of unity) the integrand into pieces where the denominator is roughly constant, we only need consider expressions

$$\int_{\mathbb{R}^n} f(y) e^{\pm i\lambda|x-y|} a(x,y) dy$$

where a is smooth and compactly supported.

Via a Taylor series expansion,

$$\lambda|x-y| = \lambda|x| - \frac{\lambda}{|x|} x \cdot y + O(1)$$

if $|x| \approx 1$, $|y| \leq \lambda^{-1/2}$

So,

$$\begin{aligned}\int_{\mathbb{R}^n} f(y) e^{i\lambda|x-y|} a(x,y) dy &= e^{i\lambda|x|} \int_{\mathbb{R}^n} e^{-i\frac{\lambda}{|x|}x\cdot y + O(1)} a(x,y) dy \\ &\approx \hat{f}\left(\lambda \frac{x}{|x|}\right) \\ &= \hat{f}(x)\Big|_{\text{Sphere of radius } \lambda}\end{aligned}$$

which is then naturally related to the restriction operator.

In the above case of our hypersurface being a sphere, Bochner-Riesz implied the Fourier Restriction conjecture. For a paraboloid, we get the opposite implication too (result of Tony Carbery), for an appropriate definition of Bochner-Riesz on a paraboloid.