LECTURE 9

SAMANTHA HOMFRAY

Given a function

$$f(x) = \sum_{n\in\mathbb{Z}^2} a_n e^{2\pi i n\cdot x} \in L^2(\mathbb{T}^2)$$

we examine solutions to the Schrodinger equation

$$\begin{cases} 2\pi i \partial_t u = \Delta_x u \\ u(x,0) = f(x) \end{cases}$$

which are given by

$$u(x,t) = \sum_{n\in\mathbb{Z}^2} a_n e^{2\pi i (n\cdot x + |n|^2 t)}$$

as can be directly verified. In the continuous case (i.e., swapping $\mathbb{T}$ with $\mathbb{R}$), the solution would have been

$$u(x,t) = \int_{\mathbb{R}^2} \hat{f}(\xi) e^{2\pi i (x\cdot\xi + |\xi|^2 t)} d\xi = \mathcal{E}\hat{f}(x,t)$$

so from the Tomas-Stein theorem, we would get a bound $||u||_{L^4(\mathbb{R}_x^2\times\mathbb{R}_t)} \lesssim ||f||_{L^2(\mathbb{R}_x^2)}$.

In the discrete case, the solution is periodic in t, so we might wonder is there a bound

$$||u(x,t)||_{L^4(\mathbb{T}_x^2\times\mathbb{T}_t)} \lesssim ||f||_{L^2(\mathbb{T}^2)}?$$

Unfortunately, there is not. The reason for this is essentially the concentration of the solution cannot disperse to infinity on a compact manifold.

The best we can do in this situation is examine the asymptotic behaviour of $||u||_{L^4(\mathbb{T}_x^2\times\mathbb{T}_t)}/||f||_{L^2(\mathbb{T}^2)}$ when f has compact Fourier support (and is thus smooth), as the support increases to infinity: Suppose

$$f(x) = \sum_{n\in\mathbb{Z}^2,\quad |n|\le N} a_n e^{2\pi i n\cdot x}$$

for a fixed N.

We analyse this in three different ways:

(1) Decoupling:

$$\begin{aligned}
||u(x,t)||^4_{L^4(\mathbb{T}^2_x\times\mathbb{T}_t)} &= \int_{[0,1]^3}\left|\sum_{n\in\mathbb{Z}^2,\quad |n|\le N} a_n e^{2\pi i(n\cdot x+|n|^2 t)}\right|^4 dxdt \\
&= \frac{1}{N^4}\int_{[0,N^2]}\int_{[0,N]^2}\left|\sum_{n\in\mathbb{Z}^2,\quad |n|\le N} a_n e^{2\pi i\left(\frac{n}{N}\cdot\widetilde{x}+\left|\frac{n}{N}\right|^2\widetilde{t}\right)}\right|^4 d\widetilde{x}d\widetilde{t} \\
&= \frac{1}{N^6}\int_{[0,N^2]^3}\left|\sum_{n\in\mathbb{Z}^2,\quad |n|\le N} a_n e^{2\pi i\left(\frac{n}{N}\cdot\widetilde{x}+\left|\frac{n}{N}\right|^2\widetilde{t}\right)}\right|^4 d\widetilde{x}d\widetilde{t}
\end{aligned}$$

since the integrand is N periodic in $\widetilde{x}$ in each direction.

Now let

$$u_n(x,t) = \frac{1}{N^{3/2}}1_{[0,N^2]^3}(x,t)a_n e^{2\pi i\left(\frac{n}{N}\cdot x+\frac{|n|^2}{N^2}t\right)}$$

so that

$$||u(x,t)||_{L^4(\mathbb{T}^2_x\times\mathbb{T}_t)} = \left(\int_{\mathbb{R}^3}\left|\sum_{n\in\mathbb{Z}^2,\quad |n|\le N} u_n(x,t)\right|^4 dxdt\right)^{1/4}$$

The characteristic function has Fourier support of approximately $[0,\frac{1}{N^2}]^3$, and the exponential portion is supported exactly at the point $\left(\frac{n}{N},\frac{|n|^2}{N^2}\right)$. So in total, we have that approximately

$$\text{supp}(\widehat{u_n}) \subset B\left(\left(\frac{n}{N},\frac{|n|^2}{N^2}\right),\frac{1}{N^2}\right)$$

which are disjoint (up to some small errors) on the paraboloid in $\mathbb{R}^3$. We can thus apply the decoupling theorem of Bourgain and Demeter to say

$$||u(x,t)||_{L^4(\mathbb{T}^2_x\times\mathbb{T}^t)} \le C_\epsilon N^\epsilon \left(\sum_{n\in\mathbb{Z}^2,\quad |n|\le N} ||u_n(x,t)||^2_{L^4(\mathbb{R}^3)}\right)^{1/2}$$
$$= C_\epsilon N^\epsilon \left(\sum_{n\in\mathbb{Z}^2,\quad |n|\le N} |a_n|^2\right)^{1/2} = C_\epsilon N^\epsilon ||f||_{L^2(\mathbb{T}^2)}$$

so our ratio is asymptotically controlled by at least N^ϵ. Note that this argument was insensitive to dimension.

(2) Number theory:

We can split up the L^4 norm as

$$||u(x,t)||^4{}_{L^4(\mathbb{T}^2_x\times\mathbb{T}_t)} = \int_{[0,1]^3} u\cdot\bar{u}\cdot u\cdot\bar{u}\; dxdt$$
$$= \int_{[0,1]^3} \sum_{\substack{n_1,n_2,n_3,n_4\in\mathbb{Z}\\ |n_i|\le N}} a_{n_1}\overline{a_{n_2}}a_{n_3}\overline{a_{n_4}} e^{2\pi i((n_1-n_2+n_3-n_4)\cdot x+(|n_1|^2-|n_2|^2+|n_3|^2-|n_4|^2)t)} dxdt$$
$$= \sum_{\substack{n_1,n_2,n_3,n_4\in\mathbb{Z}\\ |n_i|\le N\\ n_1+n_3=n_2+n_4\\ |n_1|^2+|n_3|^2=|n_2|^2+|n_4|^2}} a_{n_1}\overline{a_{n_2}}a_{n_3}\overline{a_{n_4}}$$

since

$$\int_{[0,1]} e^{2\pi ikt}dt = \begin{cases} 1, & k=0\\ 0, & k\neq 0\end{cases}$$

So the norm can be expressed in terms of some weights applied to solutions of Diophantine equations. In this case, this unfortunately makes the problem much harder instead of easier. But in general it is a useful technique, and also provides further motivation for why we would care about this bound.

(3) Geometry:

Not all is lost from the number theoretic strategy: we can rewrite the constraints on the n_i as

$$(n_1-n_2)\cdot(n_1+n_2) = (n_4-n_3)\cdot(n_4+n_3) \tag{1}$$
$$n_1-n_2 = n_4-n_3 \tag{2}$$

(as the dot product is symmetric so (1) is exactly the difference of two squares), so

$$(n_1 - n_2) \cdot (n_1 + n_2 - n_4 - n_3) = 0$$

by substituting (2) into (1) and subtracting. But (2) can be rewritten as

(3) $$n_1 - n_4 = n_2 - n_3$$

so $n_1 + n_2 - n_4 - n_3 = 2(n_2 - n_3)$, meaning

(4) $$(n_1 - n_2) \cdot (n_2 - n_3)$$

(2) and (3) imply that between the four points there are two pairs of equal distances, pointing in the same directions, and (4) shows that the directions between the ones that are not paired are orthogonal. Therefore, the four points lie on a rectangle in $\mathbb{Z}^2$.

It has been known for some time (Pach, Sharir) that for any set $S \subset \mathbb{R}^2$,

$$\#\text{rectangles in } S \leq c(\#S)^2 \log(S)$$

because for each of the $(\#S)^2$ pairs of points in S, there will be a pair of opposite points on the circle they define about $\log(S)$ of the time.

In 2023, Herr and Kwak used this to show that

$$||u(x,t)||_{L^4(\mathbb{T}^2_x \times \mathbb{T}_t)} \leq C \left(\log N\right)^{1/4} ||f||_{L^2(\mathbb{T}^2)}$$

By formally defining the dot product on finite fields, an analogous question about "rectangles" in a subset $S \subset \mathbb{F}_q^n$ can be asked, but here the best possible bound for the number of rectangles in S is $(\#S)^{2.5}$. So the structure of $\mathbb{R}^2$ was important in the above (Indeed, the original proof of Pach and Sharir used techniques from algebraic topology).

In summary, decoupling gives us an N^ϵ bound in any number of dimensions, and the geometric perspective gives a $(\log N)^{1/4}$ bound in two dimensions. Also, the problem can be realised in terms of solutions to Diophantine equations, giving a connection to number theory.

If we move to one dimension, the solution to the Schrodinger equation is

$$u(x,t)=\sum_{n=1}^{N} a_n e^{2\pi i(nx+n^2t)}$$

so we could now ask for the best $L^6(\mathbb{T}_x\times\mathbb{T}_t)$ bound of u for normalised initial data (as 6 is now the appropriate Tomas-Stein exponent). This is unknown! Some partial results are

(1) Bourgain, Demeter 2015:

$$||u(x,t)||_{L^6(\mathbb{T}_x\times\mathbb{T}_t)}\le C_\epsilon N^\epsilon\left(\sum_{n=1}^{N}|a_n|^2\right)$$

(2) Guth, Maldagne, Wang 2019:

$$||u(x,t)||_{L^6(\mathbb{T}_x\times\mathbb{T}_t)}\le C_1(\log N)^{c_2}\left(\sum_{n=1}^{N}|a_n|^2\right)$$

(3) Guo, Li, Yung: $c_2=2$ in the above

The conjectured best bound is $c_2=1/6$.

In higher dimensions, the appropriate question is what control can be put on

$$\left(\int_{[0,1]^d}\left|\sum_{n=1}^{N} a_n e^{2\pi i(nx_1+\ldots+n^dx_d)}\right|^p dx_1\ldots dx_d\right)^{1/p}$$

in terms of $\left(\sum_{n=1}^{N}|a_n|^2\right)^{1/2}$?

This is related (similarly to the two-dimensional case) to the system of Diophantine equations

$$\begin{cases} n_1+\ldots+n_{p/2}=n_{p/2+1}+\ldots+n_p\\ n_1^2+\ldots+n_{p/2}^2=n_{p/2+1}^2+\ldots+n_p^2\\ \vdots\\ n_1^d+\ldots+n_{p/2}^d=n_{p/2+1}^d+\ldots+n_p^d\end{cases}$$

for $n_i\le N$. We trivially get at least $N^{p/2}$ solutions by setting the first half of the n_i equal to the second half.

The difference between the left and right of the first equation can take on approximately N values (up to a constant factor), so the probability that a random tuple solves it is on the order of $1/N$. Similarly, the probability of solving the second equation is about $1/N^2$, and so on. So the total number of solutions should be about

$$N^p \cdot \frac{1}{N} \cdots \frac{1}{N^d} = N^{p - \frac{d(d+1)}{2}}$$

This equals $N^{p/2}$ when $p = d(d+1)$, and the two heuristics agreeing here makes this the most interesting estimate.

It is known that the ratio of these quantities at the critical exponent $p = d(d+1)$ is subject to epsilon losses (i.e., a factor of $C_\epsilon N^\epsilon$). This is due to

(1) $d = 2$: Elementary
(2) $d = 3$: Wooley (2014), using number theory techniques
(3) $d \geq 4$: Bourgain, Demeter, Guth (2015), using decoupling and multilinear Kakeya
(4) $d \geq 4$: Wooley (2018) read the above proof and realised his argument for $d = 3$ could be extended.
(5) $d \geq 4$: Guo, Li, Zorin-Kranich and Yung then translated Wooley's amended proof back to harmonic analysis, providing a simplification of Bourgain, Demeter and Guth's paper.