DECOUPLING FOR THE CONE VIA THE PRAMANIK-SEEGER ITERATION

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Below we deduce decoupling inequality for the truncated cone

$$C := \left\{ (\xi_1, \xi_2, \xi_3) \colon \xi_3 = \sqrt{\xi_1^2 + \xi_2^2},\, 1 \leq \sqrt{\xi_1^2 + \xi_2^2} \leq 2 \right\} \subset \mathbb{R}^3$$

using the Bourgain–Demeter decoupling theorem for the parabola and the Pramanik-Seeger iteration.

By rotational symmetry we focus around the point $(\xi_1, \xi_2, \xi_3) = (0, 1, 1)$ on C. We Taylor expand the defining function of C around $(0, 1, 1)$:

$$\begin{aligned}
\xi_3 - \sqrt{\xi_1^2 + \xi_2^2} &= \xi_3 - \xi_2 \sqrt{1 + \frac{\xi_1^2}{\xi_2^2}} \\
&= \xi_3 - \xi_2 \left(1 + \frac{\xi_1^2}{2\xi_2^2} + O(\frac{\xi_1^4}{\xi_2^4}) \right) \\
&= \xi_3 - \xi_2 - \frac{\xi_1^2}{2} + O(\xi_1^2 |\xi_2 - 1|) + O(\xi_1^4) \\
&= \xi_3 - \xi_2 - \frac{\xi_1^2}{2} + O\left(\xi_1^2 (|\xi_2 - 1| + \xi_1^2) \right).
\end{aligned}$$

Thus it will be convenient to perform a linear change of coordinates:

$$(\eta_1, \eta_2, \eta_3) = T(\xi_1, \xi_2, \xi_3) := (\xi_1, \xi_2, \xi_3 - \xi_2),$$

so that the point $(\xi_1, \xi_2, \xi_3) = (0, 1, 1)$ on C corresponds to $(\eta_1, \eta_2, \eta_3) = T(0, 1, 1) = (0, 1, 0)$ on $T(C)$, and

$$T(C) = \left\{ (\eta_1, \eta_2, \eta_3) \colon \eta_3 = \sqrt{\eta_1^2 + \eta_2^2} - \eta_2,\, 1 \leq \sqrt{\eta_1^2 + \eta_2^2} \leq 2 \right\}.$$

Around the point $(\eta_1, \eta_2, \eta_3) = (0, 1, 0)$, the defining function for $T(C)$ is now

$$\eta_3 - \frac{\eta_1^2}{2} + O\left(\eta_1^2 (|\eta_2 - 1| + \eta_1^2) \right)$$

so if we can, it will be advantageous to consider only a neighborhood of $(0, 1, 0)$ on $T(C)$ where both $|\eta_1|$ and $|\eta_2 - 1|$ are small. This suggests one consider, for $0 < \mu, \nu \ll 1$, functions whose Fourier transform is supported in

$$\mathcal{N}_\delta^{\mu,\nu}(C) = \left\{ (\xi_1, \xi_2, \xi_3) \colon \left| \xi_3 - \sqrt{\xi_1^2 + \xi_2^2} \right| \leq \delta,\, 1 \leq \sqrt{\xi_1^2 + \xi_2^2} \leq 1 + \delta^\nu,\, \xi_2 > 0,\, \left| \arctan \frac{\xi_1}{\xi_2} \right| \leq \delta^\mu \right\},$$

so that if $\nu \leq 2\mu \leq 1 - \nu$, then

$$\begin{aligned}
T(\mathcal{N}_\delta^{\mu,\nu}(C)) &= \left\{ (\eta_1, \eta_2, \eta_3) \colon \left| \eta_3 - \sqrt{\eta_1^2 + \eta_2^2} + \eta_2 \right| \leq \delta,\, 1 \leq \sqrt{\eta_1^2 + \eta_2^2} \leq 1 + \delta^\nu,\, \eta_2 > 0,\, \left| \arctan \frac{\eta_1}{\eta_2} \right| \leq \delta^\mu \right\} \\
&\subset \left\{ (\eta_1, \eta_2, \eta_3) \colon \eta_1 = O(\delta^\mu),\, \eta_2 = 1 + O(\delta^\nu),\, \eta_3 = \frac{1}{2}\eta_1^2 + O\left(\eta_1^2 (|\eta_2 - 1| + \eta_1^2) \right) + O(\delta) \right\} \\
&\subset \left\{ (\eta_1, \eta_2, \eta_3) \colon \eta_1 = O(\delta^\mu),\, \eta_3 = \frac{1}{2}\eta_1^2 + O(\delta^{2\mu+\nu}) \right\}.
\end{aligned}$$

(The first inclusion follows since there $\eta_2 = 1 + O(\delta^{2\mu}) + O(\delta^\nu)$ and $\nu \leq 2\mu$, while the second conclusion follows since $O(\eta_1^2(|\eta_2 - 1| + \eta_1^2)) = O(\delta^{2\mu}(\delta^\nu + \delta^{2\mu})) \leq O(\delta^{2\mu+\nu})$ when $\nu \leq 2\mu$, and $O(\delta) \leq O(\delta^{2\mu+\nu})$ when $2\mu + \nu \leq 1$.) In what follows we will fix $0 < \nu \ll 1$, and gradually enlarge μ from $\mu = \nu/2$ to $\mu = (1 - \nu)/2$. Indeed, if f has Fourier transform supported in $\mathcal{N}_\delta^{\mu,\nu}(C)$, then $f(T^{-t}(x))$ has Fourier transform supported in $T(\mathcal{N}_\delta^{\mu,\nu}(C))$, and if we freeze the x_2 variable, the function $F(x_1,x_3):=f(T^{-t}(x_1,x_2,x_3))$ has Fourier transform supported in a $O(\delta^{2\mu+\nu})$ neighborhood of the parabola of length $O(\delta^\mu)$ in $\mathbb{R}^2$. The decoupling theorem for the unit parabola in $\mathbb{R}^2$ allows one to decouple F into $\delta^{-\nu/2}$ pieces down to frequency rectangles whose long side has length $O(\delta^{\mu+\frac{\nu}{2}})$. But this can be done for every fixed $x_2\in\mathbb{R}$. Thus Minkowski inequality implies that if $\widehat{f}$ is supported in $\mathcal{N}_\delta^{\mu,\nu}(C)$ with $0<\nu\le 2\mu\le 1-\nu$, then

$$\|f\|_{L^6(\mathbb{R}^3)}\le C_\varepsilon\delta^{-\frac{\nu}{2}\varepsilon}\|\|f_\tau\|_{L^6(\mathbb{R}^3)}\|_{\ell^2(\tau)}$$

where each f_τ has a Fourier support in a rotated copy of $\mathcal{N}_\delta^{\mu+\frac{\nu}{2},\nu}(C)$.

Suppose now f has Fourier transform supported in a δ neighborhood of C. A trivial decoupling (using Minkowski inequality) in the ξ_3 direction decouples f into $\delta^{-\nu}$ many pieces, with each piece Fourier supported in a rotated copy of $\mathcal{N}_\delta^{0,\nu}(C)$. Another trivial decoupling decouples each of these $\delta^{-\nu}$ many pieces further into $\delta^{-\nu/2}$ many pieces, with each piece Fourier supported in a rotated copy of $\mathcal{N}_\delta^{\frac{\nu}{2},\nu}(C)$. If we had initially fixed $\nu=1/M$ for some big positive integer M, then after iterating the above inequality $M-1$ times from $\mu=\frac{\nu}{2}$ to $\mu=(M-1)\frac{\nu}{2}$, we obtain

$$\|f\|_{L^6(\mathbb{R}^3)}\le\delta^{-\nu}\delta^{-\nu/2}C_\varepsilon^{M-1}\delta^{-(M-1)\frac{\nu}{2}\varepsilon}\|\|f_\theta\|_{L^6(\mathbb{R}^3)}\|_{\ell^2(\theta)}$$

where $\{\theta\}$ are $1\times\delta^{1/2}\times\delta$ caps that tile a δ neighborhood of C. It remains to note $(M-1)\nu\le 1$, and choose $\nu\le\varepsilon/3$ so that

$$\|f\|_{L^6(\mathbb{R}^3)}\lesssim_\varepsilon\delta^{-\varepsilon}\|\|f_\theta\|_{L^6(\mathbb{R}^3)}\|_{\ell^2(\theta)}.$$