Math 350 Fall 2011
Homework 1

1. Which of the following are examples of groups? Among those that are groups, which of them are abelian? Explain.
   (a) The set of all real numbers under usual addition of real numbers
   (b) The set of all real numbers under usual multiplication of real numbers
   (c) The set of all non-zero real numbers under usual multiplication of real numbers
   (d) The set of all functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ under composition
   (e) The set of all bijective functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ under composition
   (f) The set of all invertible n by n matrices under matrix multiplication
2. Which of the following are examples of fields? Explain.
   (a) The set of all real numbers under usual addition and multiplication
   (b) The set of all rational numbers under usual addition and multiplication
   (c) The set of all invertible 2 by 2 matrices under usual matrix addition and multiplication
   (d) The set of all real numbers of the form $a+b\sqrt{3}$, where $a, b \in \mathbb{Q}$, under usual addition and multiplication of real numbers
   (e) $F_3 = \{0, 1, 2\}$, where the addition and multiplication were defined modulo 3 (i.e. defined as in class)
3. Let's check whether the set of all integers form a field under usual addition and multiplication of integers.
   (a) Is $(\mathbb{Z}, +)$ an abelian group? If yes, what is the additive identity?
   (b) Consider $(\mathbb{Z} \setminus \{0\}, \times)$. If this were a group, what would the multiplicative identity be?
   (c) Is there a multiplicative inverse for 2 in $(\mathbb{Z} \setminus \{0\}, \times)$?
   (d) Is $(\mathbb{Z}, +, \times)$ a field?
4. Let $\mathbb{Q}[\sqrt{2}]$ be the set of all real numbers of the form $a + b\sqrt{2}$, where $a, b \in \mathbb{Q}$. Show that this set is closed under multiplication of real numbers, i.e. the product of any two elements of $\mathbb{Q}[\sqrt{2}]$ is still in $\mathbb{Q}[\sqrt{2}]$.
5. Find the multiplicative inverse of $3 + \sqrt{2}$ in the field $\mathbb{Q}[\sqrt{2}]$. You should express your answer in the form $a + b\sqrt{2}$, where $a, b \in \mathbb{Q}$.
6. Show that every non-zero element in the field $F_7 = \{0, 1, 2, 3, 4, 5, 6\}$ has a multiplicative inverse, where the multiplication on F_7 is defined modulo 7. (Hint: You may compute the inverses one after another.)
7. Show that if F is a field and 0 is the additive identity of F, then $0 \cdot x = 0$ for all $x \in F$.
8. Show that if F is a field and 1 is the multiplicative identity of F, then $x+(-1)\cdot x = 0$ for all $x \in F$.
9. If F is a field and 1 is the multiplicative identity of F, we define $2 := 1 + 1$. Is there a field where 2 does not have a multiplicative inverse? In other words, is there a field where one cannot 'divide by 2'? If yes, give an example.
10. Suppose F is a field. Suppose 1 is the multiplicative identity of F, and 0 is the additive identity of F.
    (a) Show that $1 + 1 + 1 + 1 = (1 + 1) \cdot (1 + 1)$.

(b) Hence, show that if $1 + 1 + 1 + 1 = 0$, then $1 + 1 = 0$. (This says that no field can be of characteristic 4.)