**Math 350 Fall 2011**
**Homework 4 Supplementary Exercises**

1. Let $P_4$ be the vector space over $\mathbb{R}$ that consists of all real polynomials on $\mathbb{R}$ of degree at most 4, and let $T\colon P_4 \to P_4$ be a linear map such that $T(x^4) = 12x^2$, $T(x^3) = 6x$, $T(x^2) = 2$ and $T(x) = T(1) = 0$.
   (a) Compute $T(ax^4 + bx^3 + cx^2 + dx + e)$ for all $a, b, c, d, e \in \mathbb{R}$.
   (b) Show that $T(p) = \frac{d^2p}{dx^2}$ for all polynomials $p \in P_4$.
   (c) Compute the kernel of $T$ and the image of $T$. (Hint/answer: the kernel of $T$ is the space of all real linear polynomials on $\mathbb{R}$. The image of $T$ is the space of all real quadratic linear polynomials on $\mathbb{R}$.)
2. Let $V$ be the vector space over $\mathbb{R}$ that consists of all $2 \times 2$ matrices with real coefficients, and let $W = \mathbb{R}$ be the 1-dimensional vector space over $\mathbb{R}$. Let $T\colon V \to \mathbb{R}$ be the map such that $T(A) = \text{tr}(A)$, where tr denotes the trace of the matrix $A$.
   (a) Is $T$ a linear map?
   (b) Compute the kernel of $T$ and the image of $T$. (Hint/answer: the kernel of $T$ is
   $$\left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$
   The image of $T$ is $\mathbb{R}$.)
3. Prove that if $T\colon V \to W$ is a linear map, then $T(0) = 0$.
4. Show that if $T\colon V \to W$ is linear, then the kernel of $T$ is a subspace of $V$.
5. Show that if $T\colon V \to W$ is linear, then the image of $T$ is a subspace of $W$.
6. Let $V$ be the vector space of all real $2 \times 2$ matrices over $\mathbb{R}$.
   (a) Find a linear map $T\colon V \to V$ such that the kernel of $T$ is the set of all symmetric $2 \times 2$ matrices. Hence conclude that the set of all symmetric $2 \times 2$ matrices is a vector space over $\mathbb{R}$.
   (b) Repeat part (a) where symmetric matrices are replaced by skew-symmetric matrices.
7. (a) Show that a linear map $T\colon V \to W$ is injective if and only if the kernel of $T$ is $\{0\}$.
   (b) Use part (a) to show that the map $T\colon \mathbb{R}^2 \to \mathbb{R}^2$, given by
   $$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$
   is injective. (Hint: One should check that this $T$ is linear to apply (a).)
8. Write down an isomorphism from the vector space of all $2 \times 2$ matrices to $\mathbb{R}^4$.
9. Write down an isomorphism from the vector space of all $2\times2$ symmetric matrices to $\mathbb{R}^3$.