Math 350 Fall 2011

Final Exam review

In this set of problems, $M_{m\times n}(F)$ is the set of $m \times n$ matrices with entries in a field F, and $P_n(F)$ is the set of all polynomials of degree $\leq n$ of one variable with coefficients in F. These are vector spaces over F. The transpose of a matrix A is denoted A^t, and the conjugate transpose of A is denoted A^*.

1. Let V be the vector space of all functions on $\mathbb{R}$. If α_1, α_2 are distinct real numbers, show that $\{e^{\alpha_1 x}, e^{\alpha_2 x}\}$ is a linearly independent subset of V.
2. What can you say about a square matrix A whose determinant is zero? List as many consequences as you can.
3. When is a system of equations $Ax = b$ solvable? List as many sufficient conditions as you can.
4. Let $C[0,\pi]$ be the real vector space that consists of all continuous functions on $[0,\pi]$. Let
$$\langle f, g\rangle = \int_0^\pi f(x)g(x)dx$$
be a real inner product on $C[0,1]$, and let $\|\cdot\|$ be the associated norm. Let $f(x) = x$ and $g(x) = \cos(3x)$. Compute:
(a) $\langle f, g\rangle$, $\|f\|$ and $\|g\|$;
(b) the distance between f and g;
(c) the angle between f and g.
5. Show that if V is an inner product space and $\{w_1, \dots, w_k\}$ is an orthogonal set in V. Suppose further that none of $w_1, \dots, w_k$ is zero. Show that
(a) $\{w_1, \dots, w_k\}$ is linearly independent;
(b) $\|w_1 + \cdots + w_k\|^2 = \|w_1\|^2 + \cdots + \|w_k\|^2$;
(c) If $v \in V$, then
$$\|v\|^2 \geq \frac{|\langle v, w_1\rangle|^2}{\langle w_1, w_1\rangle} + \cdots + \frac{|\langle v, w_k\rangle|^2}{\langle w_k, w_k\rangle}.$$
6. Let $T\colon P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear map
$$T(p(x)) = p''(x) + p'(x) + p(x).$$
(a) Find the kernel and the range of T.
(b) Find all eigenvalues of T, and their corresponding eigenvectors.
(c) What is the algebraic and geometric multiplicity of each eigenvalue of T? Is T diagonalizable over $\mathbb{R}$?
7. Suppose V is a vector space over a field F, and $T\colon V \to V$ is a linear map. Suppose also that $v \in V$, and $T^{k+1}v = 0$ for some positive integer k. If $T^k v \neq 0$, show that
(a) $\{v, Tv, T^2v, \dots, T^k v\}$ is linearly independent;
(b) the span of $\{v, Tv, T^2v, \dots, T^k v\}$ is a T-invariant subspace of V.

8. Let V be the vector space $M_{2\times 2}(\mathbb{R})$ over $\mathbb{R}$.
   (a) Show that
   $$\langle A, B\rangle = \operatorname{trace}(B^t A)$$
   defines a real inner product on V. We use this inner product on V for the sequel of the question.
   (b) Write down an orthonormal basis of V.
   (c) Suppose W is the subspace of V consisting of all symmetric matrices.
      (i) Find an orthonormal basis of W.
      (ii) Find the orthogonal projection of the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ onto W.
      (iii) Let $W^\perp$ be the orthgonal complement of W in V. Show that $W^\perp$ is the set of all skew-symmetric matrices in V.
      (iv) Show that the orthogonal projection of A onto W is given by $\frac{1}{2}(A+A^t)$ for all $A \in V$. (Hint: Every $A \in V$ can be written as
      $$A = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t),$$
      and we have $\frac{1}{2}(A+A^t) \in W$, $\frac{1}{2}(A-A^t) \in W^\perp$ for all $A \in V$. One can then invoke the result in part (b) of the next question.)
      (v) Use part (iv) to check your answer in part (ii).
9. Suppose V is an inner product space, and W is a finite dimensional subspace of V. Let $W^\perp$ be the orthogonal complement of W in V.
   (a) Show that if $w \in W \cap W^\perp$, then $w = 0$.
   (b) Show that if $v \in V$ can be written as $v = w_1 + w_2$, where $w_1 \in W$ and $w_2 \in W^\perp$, then w_1 is the orthogonal projection of v onto W.
   (c) Show that if in addition V is finite dimensional, then $(W^\perp)^\perp = W$.
   (d) Hence, or otherwise, show that if V is finite dimensional, then under the assumptions of part (b), w_2 is the orthogonal projection of v onto $W^\perp$.
10. Let $\mathfrak{so}(3)$ be the set of all skew-symmetric 3×3 complex matrices, i.e.
   $$\mathfrak{so}(3) = \{A \in M_{3\times 3}(\mathbb{C})\colon A^t = -A\}.$$
   (a) Show that $\mathfrak{so}(3)$ is a vector space over $\mathbb{C}$.
   (b) Show that
   $$\left\{ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}$$
   is a basis of $\mathfrak{so}(3)$. Hence determine the dimension of $\mathfrak{so}(3)$.
   (c) Given two square matrices A and B, we define $[A, B]$ to be the matrix $AB - BA$. Show that $[A, B] \in \mathfrak{so}(3)$ for all $A, B \in \mathfrak{so}(3)$.

11. We continue from the previous question. Let

$$H = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

   (a) Show that $H \in \mathfrak{so}(3)$.
   (b) Let $T\colon \mathfrak{so}(3) \to \mathfrak{so}(3)$ be defined by $T(A) = [H, A]$ for all $A \in \mathfrak{so}(3)$.
      (i) Is T a linear map? Explain.
      (ii) If α is the ordered basis given in part (b) of the previous question, compute the matrix representation $[T]_\alpha$ of T with respect to α.
      (iii) Show that T is diagonalizable over $\mathbb{C}$. (Hint: You can do this without finding all the eigenvectors of T.)
12. Show that if A is a 2×2 matrix, then $A^2 = s_1 A - s_2 I$ where $s_1 = \text{trace}(A)$ and $s_2 = \det(A)$. Here I is the 2×2 identity matrix. (Hint: Use Cayley-Hamilton.)
13. Let B be an $n \times n$ matrix with entries in $\mathbb{R}$. Let $A = B^t B$, and $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{R}^n$. Show that
   (a) A is real symmetric, in the sense that $A^t = A$;
   (b) $\langle Ax, x \rangle \geq 0$ for all $x \in \mathbb{R}^n$;
   (c) If λ is an eigenvalue of A, then $\lambda \geq 0$.
14. Given two vectors $v_1, v_2 \in \mathbb{R}^3$, we define a map $T\colon \mathbb{R}^3 \to \mathbb{R}$ as follows: for any $v \in \mathbb{R}^3$, let $A(v)$ be the 3×3 matrix whose first row is v, second row is v_1 and third row is v_2. We then define $T(v) = \det(A(v))$.
   (a) Show that T is a linear map.
   (b) Equip $\mathbb{R}^3$ with the standard inner product $\langle \cdot, \cdot \rangle$. Then the Riesz representation theorem says that there exists a unique vector $w \in \mathbb{R}^3$ such that $T(v) = \langle v, w \rangle$ for all $v \in \mathbb{R}^3$. Show that this w is given by the cross product of v_1 with v_2.

   (One can generalize this exercise and define the 'cross product' of $n-1$ vectors in $\mathbb{R}^n$ for all n.)
15. Suppose A is a complex $n \times n$ matrix.
   (a) Show that if λ is an eigenvalue of A, then $\overline{\lambda}$ is an eigenvalue of A^*.
   (b) More generally, show that $\dim(\text{nullspace}(A - \lambda I)) = \dim(\text{nullspace}(A^* - \overline{\lambda} I))$ for all $\lambda \in \mathbb{C}$.
   (c) (Optional) Show that if in addition A is normal, i.e. if $AA^* = A^*A$, then not only the dimensions of the above nullspaces are equal, but

$$\text{nullspace}(A - \lambda I) = \text{nullspace}(A^* - \overline{\lambda} I)$$

   for all $\lambda \in \mathbb{C}$.
   (d) Show that the conclusion of (c) is not true if A is not normal. (Hint: consider upper triangular matrices A that are not normal.)

16. The goal of this question is to illustrate some techniques in computing determinants. For each $x \in \mathbb{R}$, let

$$A_x = \begin{pmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{pmatrix}.$$

We want to compute $f(x) = \det(A_x)$.

(a) Show that $f(x)$ is a cubic polynomial in x, and the coefficient of x^3 is 1.

(b) Show that $f(x) = 0$ when $x = 1$. Hence show that $x - 1$ is a factor of $f(x)$.

(c) Show that

$$f'(x) = \det \begin{pmatrix} 1 & 0 & 0 \\ 1 & x & 1 \\ 1 & 1 & x \end{pmatrix} + \det \begin{pmatrix} x & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & x \end{pmatrix} + \det \begin{pmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 0 & 0 & 1 \end{pmatrix} = 3 \det \begin{pmatrix} x & 1 \\ 1 & x \end{pmatrix}.$$

Hence show that $f'(x) = 0$ when $x = 1$, and that $(x-1)^2$ is a factor of $f(x)$.

(d) Show that $f(x) = 0$ when $x = -2$, and hence $x + 2$ is a factor of $f(x)$.

(e) Together, conclude that $f(x) = (x-1)^2(x+2)$.

17. Incidentally, the above problem allows us to further analyze the matrix A_x.

(a) Show that the eigenvalues of A_x are $x - 1$ and $x + 2$.

(b) Show that A_x is diagonalizable for all x. (Hint: A_x is symmetric for all x.)

(c) Show that $(A_x - (x-1)I)(A_x - (x+2)I)v = 0$ for all $x \in \mathbb{R}$ and all $v \in \mathbb{R}^3$, where I is the 3×3 identity matrix.

(d) Show that $A_x^2 - (2x+1)A_x + (x-1)(x+2)I = 0$ for all x. Conclude that if $x \neq 1$ or -2, then A_x is invertible, with

$$A_x^{-1} = -\frac{1}{(x-1)(x+2)}(A_x - (2x+1)I) = -\frac{1}{(x-1)(x+2)}A_{-x-1}.$$

18. The technique in Question 16 can be used to compute the following Vandermonde determinant:

$$f(\alpha_1, \dots, \alpha_n) := \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \alpha_1^2 & \alpha_2^2 & \dots & \alpha_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \dots & \alpha_n^{n-1} \end{pmatrix}.$$

(a) Show that $f(\alpha_1, \dots, \alpha_n)$ is a polynomial of $\alpha_1, \dots, \alpha_n$, and that $f(\alpha_1, \dots, \alpha_n) = 0$ whenever $\alpha_i = \alpha_j$ for some $1 \leq i < j \leq n$. Hence $\prod_{1 \leq i < j \leq n}(\alpha_j - \alpha_i)$ divides $f(\alpha_1, \dots, \alpha_n)$.

(b) Conclude that

$$f(\alpha_1, \dots, \alpha_n) = \prod_{1 \leq i < j \leq n} (\alpha_j - \alpha_i).$$

19. (a) Let V be a vector space over $\mathbb{R}$ and let $T\colon V \to V$ be a linear operator. Let $v_1, \dots, v_k$ be eigenvectors of T corresponding to eigenvalues $\lambda_1, \dots, \lambda_k$, and suppose $\lambda_1, \dots, \lambda_k$ are pairwise distinct. Prove that $\{v_1, \dots, v_k\}$ are linearly independent.

    (b) Let V be the vector space of all infinitely differentiable functions on $\mathbb{R}$. If $\alpha_1, \dots, \alpha_n$ is a list of distinct real numbers, show that $\{e^{\alpha_1 x}, e^{\alpha_2 x}, \dots, e^{\alpha_n x}\}$ is a linearly independent subset of V. This is a generalization of Question 1. (Hint: One may use part (a) (how?), or the result of the previous question; in fact, if $c_1 e^{\alpha_1 x} + \cdots + c_n e^{\alpha_n x} = 0$ for some scalars $c_1, \dots, c_n$, then by evaluating the k-th derivative of both sides at $x = 0$, we get

$$c_1 \alpha_1^k + \cdots + c_n \alpha_n^k = 0 \quad \text{for all } k = 0, 1, 2 \dots.$$

In particular,

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \alpha_1^2 & \alpha_2^2 & \dots & \alpha_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \dots & \alpha_n^{n-1} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since $\alpha_1, \dots, \alpha_n$ are assumed pairwise distinct, the coefficient matrix of this system of linear equations is non-zero by the result of the previous question. Thus this system of linear equations has no non-trivial solutions, i.e. $c_1 = c_2 = \cdots = c_n = 0$.)

20. Suppose $\mathcal{P}_k$ is the set of all polynomials on $\mathbb{R}^2$ that are homogeneous of degree k, i.e. $\mathcal{P}_k = \left\{ \sum_{j=0}^k a_j x^j y^{k-j} \colon a_0, \dots, a_k \in \mathbb{R} \right\}$ for all $k \in \mathbb{N}$. It is easy to see that $\mathcal{P}_k$ is a vector space over $\mathbb{R}$.

    For each $P \in \mathcal{P}_k$, we define a differential operator

$$P(D) := \sum_{j=0}^k a_j \left(\frac{\partial}{\partial x} \right)^j \left(\frac{\partial}{\partial y} \right)^{k-j} \quad \text{if } P(x,y) = \sum_{j=0}^k a_j x^j y^{k-j}.$$

For $P, Q \in \mathcal{P}_k$, define

$$\langle P, Q \rangle = P(D)(Q(x,y)).$$

(a) Show that this defines a real inner product on $\mathcal{P}_k$.

(b) Let W be the subspace of V given by

$$W = \{ R \in \mathcal{P}_k \colon R(x,y) = (x^2 + y^2) Q(x,y) \text{ for some } Q(x,y) \in \mathcal{P}_{k-2} \}.$$

Show that the orthogonal complement of W in $\mathcal{P}_k$ is the space of homogeneous harmonic polynomials of degree k, i.e.

$$W^\perp = \left\{ P \in \mathcal{P}_k \colon \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 0 \right\}.$$

21. This is a continuation of Question 20 of the review for midterm 2. Let $\mathfrak{gl}(n) = M_{n\times n}(\mathbb{R})$, and $\mathfrak{sl}(2)$ be the subspace of $\mathfrak{gl}(2)$ that consists of matrices whose trace is zero. These are vector spaces over $\mathbb{R}$. Let X, Y, H be elements of $\mathfrak{sl}(2)$, given by

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let N be a positive integer, and $T\colon \mathfrak{sl}(2) \to \mathfrak{gl}(N+1)$ be a linear map satisfying

$$T([A,B]) = [T(A), T(B)] \quad \text{for all } A, B \in \mathfrak{sl}(2).$$

(Recall that $[A,B] = AB - BA$ if A and B are two square matrices.) Let also

$$V_\lambda = \{v \in \mathbb{R}^{N+1} \colon T(H)v = \lambda v\} \quad \text{for all } \lambda \in \mathbb{R}.$$

Show that

(a) V_λ is the eigenspace of $T(H)$ with eigenvalue λ if λ is an eigenvalue of $T(H)$, and $\{0\}$ otherwise.

(b) If $\lambda \in \mathbb{R}$ and $v \in V_\lambda$, then $T(H)v \in V_\lambda$, $T(X)v \in V_{\lambda+2}$, and $T(Y)v \in V_{\lambda-2}$.

(c) Suppose from now on that the linear map T satisfies the following two properties:

- $T(H)$ is diagonalizable as a linear map from $\mathbb{R}^{N+1}$ to $\mathbb{R}^{N+1}$;
- If W is a subspace of $\mathbb{R}^{N+1}$ that is both $T(X)$-invariant, $T(Y)$-invariant and $T(H)$-invariant, then $W = \{0\}$ or $W = \mathbb{R}^{N+1}$.

Suppose λ_0 is the largest eigenvalue of $T(H)$, and v_0 is an eigenvector of $T(H)$ with eigenvalue λ_0.

(i) Let W be the subspace of $\mathbb{R}^{N+1}$ spanned by

$$\{v_0, T(Y)v_0, [T(Y)]^2 v_0, \dots\}.$$

Show that W is $T(X)$-invariant, $T(Y)$-invariant and $T(H)$-invariant. Thus conclude that $W = \mathbb{R}^{N+1}$.

(ii) Let k be the smallest integer such that $T(Y)^{k+1}(Y) = 0$. Show that

$$\{v_0, T(Y)v_0, [T(Y)]^2 v_0, \dots, [T(Y)]^k v_0\}$$

is linear independent. Hence show that it is a basis of W, and conclude that $k = N$.

(iii) Show that

$$0 = T(X)[T(Y)]^{N+1} v_0 = (N+1)(\lambda_0 - N)[T(Y)]^N v_0.$$

Hence conclude that $\lambda_0 = N$, and that the eigenvalues of $T(H)$ are $N, N-2, N-4, \dots, -N$.

(iv) Show that V_m is one-dimensional for all $m = N, N-2, N-4, \dots, -N$, and is $\{0\}$ otherwise.