Math 350 Fall 2011
Midterm 1 review

1. Which of the following are subspaces of $\mathbb{R}^n$? Prove your assertion.
   (a) $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 + 2x_2 + 3x_3 + \cdots + nx_n = 0\}$
   (b) $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 + 2x_2 + 3x_3 + \cdots + nx_n = 1\}$
   (c) $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 = x_2 = \cdots = x_n\}$
   (d) $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + x_2^2 + \cdots + x_n^2 = 1\}$
   (e) $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + x_2^2 + \cdots + x_n^2 = 0\}$

2. Suppose V is the set of all polynomials on $\mathbb{R}$ with real coefficients,
$$V_n = \{P \in V : \deg P \leq n\},$$
and
$$W = \{P \in V : P(x) = -P(-x) \text{ for all } x\}.$$
The usual addition and scalar multiplication for polynomials will be used throughout the question.
   (a) Prove that V is a vector space over $\mathbb{R}$. (Hint: Check all axioms of a vector space.)
   (b) Prove that V_n is a subspace of V for all $n \in \mathbb{N} \cup \{0\}$.
   (c) Is V_n a vector space over $\mathbb{R}$? (Hint: You can use parts (a) and (b).)
   (d) Is W a vector space over $\mathbb{R}$? (Hint: You may mimic your strategy in (c).)

3. Let V be the following vector spaces over $\mathbb{R}$, and S be the following subsets of V. Is S linearly independent? Does S span V? Prove your assertions.
   (a) $V = \mathbb{R}^3$, $S = \{(1, 2, 3), (0, 1, 2), (1, 0, 3)\}$
   (b) $V = \mathbb{R}^3$, $S = \{(1, 2, 3), (0, 1, 2), (-1, 0, 1)\}$
   (c) $V = \mathbb{R}^3$, $S = \{(1, 2, 3), (0, 1, 2), (-1, 0, 1), (1, 1, 1)\}$
   (d) $V = \mathbb{R}^3$, $S = \{(1, 2, 3), (4, 5, 6)\}$
   (e) $V = \{2\times2 \text{ real matrices}\}$, $S = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}$
   (f) $V = \{2\times2 \text{ real matrices}\}$, $S = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}\}$
   (g) $V = \{$ polynomials on $\mathbb{R}$ with real coefficients and that has degree at most 3 $\}$,
   $S = \{1 + x + x^2, x + x^2 + x^3, 1 + x^3, x - x^2\}$
   (h) $V = \{$ even polynomials on $\mathbb{R}$ with real coefficients $\}$, $S = \{1, x^2, x^4, x^6, \ldots\}$
   (i) $V = \{$ polynomials on $\mathbb{R}$ with real coefficients $\}$,
   $S = \{1, (x - 1), (x - 1)^2, (x - 1)^3, \ldots\}$

4. Let $V = \{$ polynomials on $\mathbb{R}$ with real coefficients and that has degree at most 3 $\}$ be a vector space over $\mathbb{R}$, $S = \{1 - x, x - x^2, x^2 - x^3\}$. Show that the span of S is given by
$$\{a_0 + a_1x + a_2x^2 + a_3x^3 : a_0 + a_1 + a_2 + a_3 = 0\}.$$

5. Write down your favorite (non-trivial) vector space, and find a basis for that vector space.

6. Write down your favorite infinite dimensional vector space, and find a basis for that vector space.

7. Suppose V is a vector space over $\mathbb{R}$, $\{v_1, \dots, v_n\} \subseteq V$, and
$$u = v_1 + 2v_2 + \cdots + (n-1)v_{n-1}.$$
   (a) Suppose $\{v_1, \dots, v_n\}$ is linearly independent.
      (i) Show that $\{u, v_2, \dots, v_n\}$ is also linearly independent.
      (ii) What about $\{v_1, u, v_3, \dots, v_n\}$ and $\{v_1, \dots, v_{n-1}, u\}$? Are they also linearly independent?
   (b) Show that the span of $\{v_1, \dots, v_n\}$ is the same as the span of $\{u, v_2, \dots, v_n\}$.

8. Let l^1 be the space of all absolutely summable real sequences, i.e.
$$l^1 = \{(x_1, x_2, \dots) \colon x_n \in \mathbb{R} \text{ for all } n = 1, 2, \dots, \text{ and } \sum_{n=1}^{\infty} |x_n| < \infty\}.$$
   (a) Is l^1 a vector space over $\mathbb{R}$? Why?
   (b) For each $j \in \mathbb{N}$, let e_j be the sequence whose j-th term is 1 and all other terms are zero. Let $S = \{e_1, e_2, \dots\}$. Is S linearly independent? Why?
   (c) Consider the sequence $(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots)$ be a sequence whose jth term is $\frac{1}{2^j}$. Is this sequence in l^1? If yes, is it in the span of S? Why or why not?
   (d) Let c_0 be the set of sequences defined by
$$c_0 = \{\{(x_1, x_2, \dots) \colon x_n \in \mathbb{R}\ \forall n = 1, 2, \dots, \text{ and } \exists N \in \mathbb{N} \text{ such that } x_n = 0\ \forall n \geq N\}.$$
      (i) Is c_0 a subset of l^1?
      (ii) Is c_0 a vector space over $\mathbb{R}$?
      (iii) Write down a basis for c_0. You should check that the set you write down is a basis.
      (iv) Is c_0 finite dimensional?

9. Let $V = \{a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} \colon a, b, c, d \in \mathbb{Q}\}$.
   (a) Check that V is a vector space over $\mathbb{Q}$. (Hint: Use $V \subseteq \mathbb{R}$, and $\mathbb{R}$ is a vector space over $\mathbb{Q}$.)
   (b) Show that $\{1, \sqrt{2}\}$ is a linearly independent subset of V over $\mathbb{Q}$.
   (c) Show that $\{1, \sqrt{2}, \sqrt{3}\}$ is a linearly independent subset of V over $\mathbb{Q}$. (Hint: It suffices to show that $\sqrt{3}$ is not in the span of $\{1, \sqrt{2}\}$ over $\mathbb{Q}$, and to do so, just notice that if $\sqrt{3} = a + b\sqrt{2}$ for some $a, b \in \mathbb{Q}$, then you can square both sides, and proceed...)
   (d) In this part, we are going to prove that $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$ is a linearly independent subset of V over $\mathbb{Q}$. Suppose $a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} = 0$ for some $a, b, c, d \in \mathbb{Q}$.
      (i) Show that if $d \neq 0$, then
$$\sqrt{3} = -\frac{a + b\sqrt{2}}{c + d\sqrt{2}}.$$
      (Why do you need $d \neq 0$ here?)
      (ii) Hence, show that if $d \neq 0$, then $\sqrt{3}$ is in the span of $\{1, \sqrt{2}\}$ over $\mathbb{Q}$.
      (iii) Hence conclude that $d = 0$.
      (iv) Hence conclude that $a = b = c = d = 0$.
   (e) Find the dimension of V over $\mathbb{Q}$.
   (f) (Optional) Show that V is a field. In fact it is the smallest field contained in $\mathbb{R}$ that contains both $\sqrt{2}$ and $\sqrt{3}$.