Math 350 Fall 2011
Midterm 2 review

In this set of problems, $M_{m\times n}$ is the vector space of real $m \times n$ matrices, and P_n is the vector space of all polynomials of degree $\leq n$ of one variable with real coefficients. These are vector spaces over $\mathbb{R}$.

1. (a) Is $T\colon \mathbb{R}^3 \to \mathbb{R}^2$, defined by $T(x,y,z) = (x+2y+3z, -x-y-z)$ a linear map? Explain.
   (b) Is $S\colon \mathbb{R}^3 \to \mathbb{R}^2$, defined by $T(x,y,z) = (x+y+z, xyz)$ a linear map? Explain.
   (c) Is $U\colon M_{n\times n} \to \mathbb{R}$, defined by $U(A) = \det(A)$, a linear map if $n \geq 2$? Explain.
   (d) Is $W\colon M_{n\times n} \to \mathbb{R}$, defined by $W(A) = \text{Trace}(A)$, a linear map? Explain.
2. Suppose $T\colon P_4 \to P_4$ is the linear map

$$T(p(x)) = x^2p''(x) + xp'(x) + p(x).$$

   (a) Show that T is a linear map.
   (b) Is T one-to-one? Verify your assertion.
   (c) Is T onto? Verify your assertion. (Hint: Use part (b).)
3. Suppose $T\colon M_{2\times 2} \to M_{2\times 2}$ is the linear map defined by

$$T(A) = A + A^t.$$

   (a) Find the range and kernel of T.
   (b) Find the rank and nullity of T.
   (c) Let

$$\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

   be an ordered basis of $M_{2\times 2}$. Find the matrix representation $[T]_\alpha$ of T with respect to α.
   (d) Compute the determinant of the 4×4 matrix you've found in part (c). (Hint: The answer can be read off from part (b) without any actual calculation.)
4. Suppose $T\colon \mathbb{R}^2 \to P_4$ is a linear map satisfying

$$T(1,1) = x + 5x^3 \quad \text{and} \quad T(3,5) = 1 - x^2 + x^4.$$

   (a) Compute $T(1,3)$.
   (b) Let $\alpha = \{(1,0),(0,1)\}$ be the standard basis of $\mathbb{R}^2$, and $\beta = \{1, x, x^2, x^3, x^4\}$ be the standard basis of P_4. Compute the matrix representation $[T]_\alpha^\beta$ of T with respect to α and β.
   (c) Find the kernel and range of T.
   (d) Find the rank and nullity of T. Is T one-to-one? Is T onto?

5. Suppose

$$A = \begin{pmatrix} 4 & 3 & -1 & 7 \\ 1 & 1 & 0 & 2 \\ 2 & 5 & 3 & 7 \end{pmatrix}.$$

   (a) Find a basis for the nullspace of A.
   (b) Find a basis for the column space of A.
   (c) If

$$b = \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix}$$

   for some real number t and the system of equations $Ax = b$ is solvable, find all possible value(s) of t.
   (d) For each of the values of t you found in (c), solve the equation $Ax = b$. (You should find *all* solutions of the equation.)

6. Let $T\colon P_3 \to P_3$ be the linear map defined by

$$T(p(x)) = 2p(x) + p(2)x.$$

   (a) Find all eigenvalues of T. Also, for each eigenvalue of T, find a basis of the corresponding eigenspace.
   (b) Is T diagonalizable over $\mathbb{R}$? Explain.

7. Show that if $T\colon V \to W$ is a linear map, then
   (a) $T(0) = 0$;
   (b) the range of T is a subspace of W;
   (c) the kernel of T is a subspace of V.

8. Show that if $T\colon V \to W$ is a linear map, then T is one-to-one if and only if the nullity of T is zero.

9. Show that if $T\colon V \to W$ is a linear map, and $\dim(V) = \dim(W) < \infty$, then T is one-to-one if and only if T is onto.

10. Let P be an invertible $n \times n$ real matrix. Define a linear map $T\colon M_{n\times n} \to M_{n\times n}$ by $T(A) = PAP^{-1}$ for all $A \in M_{n\times n}$. Show that T is an isomorphism.

11. Suppose $T\colon V \to W$ is a linear map between two finite dimensional vector spaces, with $\dim(V) = n$, $\dim(W) = m$.
    (a) Show that

$$\text{nullity}(T) \geq n - m.$$

    Hence conclude that T cannot be one-to-one if $n > m$.
    (b) Show that

$$\text{rank}(T) \leq \min\{n, m\}.$$

    Hence conclude that T cannot be onto if $n < m$.

12. Show that if A is an invertible $n \times n$ real matrix, then $\det(A^{-1}) = (\det(A))^{-1}$.

13. Suppose A, B are two $n \times n$ matrices with entries in $\mathbb{R}$ (or $\mathbb{C}$). Show that it is impossible that $AB - BA = I$, where I is the $n \times n$ identity matrix. (Hint: Take the trace.)
14. Suppose A is a square matrix with $A^n = 0$ for some positive integer n, and I is the identity matrix of the same size as A.
    (a) Show that $I - A$ is invertible, and
    $$(I - A)^{-1} = I + A + A^2 + \cdots + A^{n-1}.$$
    (Hint: Just compute $(I - A)(I + A + A^2 + \cdots + A^{n-1})$.)
    (b) Hence show that $\det(I - A) \neq 0$. Similarly, show that $\det(I + A) \neq 0$.
15. Suppose A is any $n \times n$ real matrix that is diagonalizable over $\mathbb{R}$. Show that
    $$\det(I + tA) = 1 + s_1 t + s_2 t^2 + \cdots + s_n t^n$$
    for some coefficients $s_1, \ldots, s_n$ that are determined by A. In fact, show that $s_1 = \text{Trace}(A)$ and $s_n = \det(A)$.
16. Suppose $T \colon V \to V$ is a bijective linear map, where V is a finite dimensional vector space over a field F. Show that
    (a) All eigenvalues of T are non-zero;
    (b) v is an eigenvector of T, if and only if v is an eigenvector of T^{-1};
    (c) λ is an eigenvalue of T, if and only if λ^{-1} is an eigenvalue of T^{-1};
    (d) T is diagonalizable, if and only if T^{-1} is diagonalizable.
17. Show that if v is an eigenvector of a linear map $T \colon V \to V$ corresponding to an eigenvalue λ, then v is an eigenvector of $T^2 \colon V \to V$ with eigenvalue λ^2. (Remember T^2 is a shorthand for $T \circ T$.)
18. (a) Show that λ is an eigenvalue of a matrix A if and only if it is an eigenvalue of A^t. (Hint: Consider $\det(A - \lambda I)$ and $\det(A^t - \lambda I)$.)
    (b) Suppose λ is an eigenvalue of a matrix A. Show that the dimension of the eigenspace of A corresponding to λ is the same as the dimension of the eigenspace of A^t corresponding to λ. (Hint: Consider the nullity of $A - \lambda I$ and the nullity of $A^t - \lambda I$.)
19. Show that if a matrix A is diagonalizable, then $\det(A)$ is the product of its eigenvalues (counting multiplicities). How does the trace of a diagonalizable matrix relate to its eigenvalues?
20. In this question, we write $\mathfrak{gl}(n)$ as a shorthand for $M_{n\times n}$. It is a vector space over $\mathbb{R}$.
    For any $A, B \in \mathfrak{gl}(n)$, we define $[A, B]$ to be the matrix given by
    $$[A, B] := AB - BA;$$
    this is sometimes called the commutator between A and B. For any $A \in \mathfrak{gl}(n)$, define a map $\text{ad}_A \colon \mathfrak{gl}(n) \to \mathfrak{gl}(n)$ by
    $$\text{ad}_A(B) = [A, B].$$

(a) Show that ad_A is a linear map for any $A \in \mathfrak{gl}(n)$.

(b) Show that for any $A, B \in \mathfrak{gl}(n)$, we have $[A, B] = -[B, A]$.

(c) Show that for any $A, B \in \mathfrak{gl}(n)$, we have $\mathrm{ad}_{[A,B]} = [\mathrm{ad}_A, \mathrm{ad}_B]$, where $[\mathrm{ad}_A, \mathrm{ad}_B]$ is by definition the linear map from $\mathfrak{gl}(n)$ to $\mathfrak{gl}(n)$ defined by $[\mathrm{ad}_A, \mathrm{ad}_B] := \mathrm{ad}_A \circ \mathrm{ad}_B - \mathrm{ad}_B \circ \mathrm{ad}_A$. (This is usually called the Jacobi identity.)

(d) Let $\mathfrak{sl}(n)$ be the space of all $n \times n$ real matrices whose trace is equal to zero. It is a subspace of $\mathfrak{gl}(n)$. Show that if $A, B \in \mathfrak{sl}(n)$, then $\mathrm{ad}_A(B) \in \mathfrak{sl}(n)$; in other words, if $A \in \mathfrak{sl}(n)$, then ad_A maps $\mathfrak{sl}(n)$ into $\mathfrak{sl}(n)$.

(e) From now on we focus our attention to the space $\mathfrak{sl}(2)$. Let

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(i) Check that

$$[X, Y] = H, \quad [H, X] = 2X, \quad [H, Y] = -2Y.$$

(ii) Check that $\{X, Y, H\}$ is a basis of $\mathfrak{sl}(2)$.

(iii) Let α be the ordered basis $\{X, Y, H\}$ of $\mathfrak{sl}(2)$. Compute the matrix representation of the linear map $\mathrm{ad}_H : \mathfrak{sl}(2) \to \mathfrak{sl}(2)$ with respect to α; i.e. compute $[\mathrm{ad}_H]_\alpha$.

(iv) Show that $\mathrm{ad}_H : \mathfrak{sl}(2) \to \mathfrak{sl}(2)$ is diagonalizable. Also find the eigenvalues of this map, and the corresponding eigenvectors.