Math 350 Fall 2011 Worksheet 1

Name:

The goal of this worksheet is to introduce to you the concept of a quotient space, and to guide you through the proof the rank-nullity theorem.

From now on, let V be a vector space over a field F, and W be a subspace of V.

Definition 1. *We say two vectors $v_1, v_2 \in V$ are equivalent*[1]*, and write $v_1 \sim v_2$, if and only if $v_1 - v_2 \in W$.*

Question 1. *For example, suppose now $V = \mathbb{R}^2$, $F = \mathbb{R}$ and $W = \{(x, 0) : x \in \mathbb{R}\}$ is the x-axis.*

(a) Can you find all vectors in V that are equivalent to the vector $(0, 0)$? Represent your answer on the Cartesian plane.

(b) Can you find all vectors in V that are equivalent to the vector $(1, 2)$? Represent your answer on the Cartesian plane.

(c) Can you find all vectors in V that are equivalent to the vector (x, y)? Represent your answer on the Cartesian plane.

[1] For those who knows equivalence relations, this is an equivalent relation on V, in the sense that

1. $v \sim v$ for all $v \in V$,
2. $v_1 \sim v_2$ if and only if $v_2 \sim v_1$, and
3. if $v_1 \sim v_2$ and $v_2 \sim v_3$, then $v_1 \sim v_3$.

So in the above example, the set of vectors in $\mathbb{R}^2$ that are equivalent to a given vector (x, y) is given by

$$\{(x, y) + (u, v) \colon (u, v) \in W\}.$$

In general, it can be shown, that if V is vector space over a field F and W is a subspace, then the set of vectors in V that are equivalent to a given vector $v \in V$ is given by

$$\{v + w \colon w \in W\}.$$

Since this set of vectors is obtained by adding to the given v an arbitrary vector $w \in W$, this set will also be written as $v + W$. In other words, we have the following definition:

Definition 2. *If v is a vector in V and W is a subspace of V, we define the set $v + W$ to be a subset of V given by*

$$v + W := \{v + w \colon w \in W\}.$$

Definition 3. *Any subset of V of the form $v + W$ for some $v \in V$ is called a* ***coset*** *of W in V, and the set of cosets of W in V is denoted as V/W.*

Let's check your understanding now. Do you understand the following statements?

(a) A coset of the x-axis in $\mathbb{R}^2$ is a line in $\mathbb{R}^2$ that is parallel to the x-axis.
(b) $\mathbb{R}^2/(x\text{-axis})$ is the set of all lines in $\mathbb{R}^2$ that is parallel to the x-axis.

Question 2. *Now let's look again at the example in Question 1, i.e. assume $V = \mathbb{R}^2$ and $W = x$-axis. We knew that the line $\{(x, y) \colon y = 2\}$ is an element of V/W. In fact this line can be written as $(1, 2) + W$, as we have seen in Question 1(b). But there are actually many ways of writing this coset in the form $v + W$ where $v \in V$. For example, the same line can also be written as $(7, 2) + W$, i.e.*

$$(1, 2) + W = (7, 2) + W.$$

(Why?) In general, check, in this example, that

$$(x_1, y_1) + W = (x_2, y_2) + W \quad \text{if and only if} \quad (x_1, y_1) - (x_2, y_2) \in W.$$

Back to our general situation. Suppose V is a vector space over a field F and W is a subspace of V. It can be shown that

Theorem 1. *Two cosets $v_1 + W$ and $v_2 + W$ are equal (as subsets of V) if and only if $v_1 - v_2 \in W$.*

The proof of this is left as an exercise.

So far we have defined V/W as a set of subsets of V. To proceed further, we are going to make V/W a vector space over F. Now V/W is a set of sets. Every element of V/W is a set of vectors from V. It is thus perhaps not completely clear how one should add two elements of V/W. We make the following definition. Take two elements from V/W. They are then of the form $v_1 + W$ and $v_2 + W$ for some $v_1, v_2 \in V$. (Why?)

Definition 4. *We define the sum of two cosets of W in V to be another coset of W in V, according to the formula*

$$(v_1 + W) + (v_2 + W) := (v_1 + v_2) + W.$$

Question 3. *Suppose we are in the example in Question 1, i.e. $V = \mathbb{R}^2$ and $W =$ x-axis. Compute $((1, 2) + W) + ((3, 7) + W)$. Draw your answer, as well as the two cosets $(1, 2) + W$ and $(3, 7) + W$, on the same Cartesian plane.*

Now the problem is that given an element of V/W, there are many ways of writing it as $v+W$ with $v \in V$. So suppose you have two elements of V/W, and you decide to write them as $v_1 + W$ and $v_2 + W$ for some particular choice of $v_1, v_2 \in V$. And suppose your friend comes in, and decides to write the same elements of V/W as $u_1 + W$ and $u_2 + W$ instead, for some different choices $u_1, u_2 \in V$. Suppose both of you add the two elements of V/W using Definition 4. Do you two get the same answer? You had better hope that this is the case, because otherwise the definition we made would not be *well-defined*. Fortunately, this is the case:

Question 4. *Prove that if*

$$v_1 + W = u_1 + W \quad \text{and} \quad v_2 + W = u_2 + W$$

for some $v_1, v_2, u_1, u_2 \in V$, then

$$(v_1 + v_2) + W = (u_1 + u_2) + W.$$

(Hint: use Theorem 1 and the properties of a subspace.)

Next, now that we have defined an addition on V/W, we proceed to define a scalar multiplication on V/W. Recall that V was a vector space over a field F. Now take an element of V/W. Again it is then a set of the form $v + W$ for some $v \in V$.

Definition 5. *If $c \in F$ and $v + W \in V/W$, we define their scalar multiplication by*

$$c \cdot (v + W) := (c \cdot v) + W.$$

Question 5. *Let's first understand what this means in the example in Question 1. If $V = \mathbb{R}^2$, $F = \mathbb{R}$ and $W =$ x-axis, compute $4 \cdot ((1, 2) + W)$. Draw both your answer, as well as the original coset $(1, 2) + W$, on the same Cartesian plane.*

Again we need to check that the scalar multiplication in Definition 5 is well-defined, because there are many ways of writing a given coset as $v + W$ for some $v \in V$. So we have

Question 6. *Prove that if*

$$v + W = u + W$$

for some $v, u \in V$, and if $c \in F$, then

$$(c \cdot v) + W = (c \cdot u) + W.$$

(Hint: use Theorem 1 and the properties of a subspace.)

Finally, we are ready to state the first main theorem today:

Theorem 2. *If V is a vector space over a field F and W is a subspace of V, then V/W is a vector space over F under the addition and scalar multiplication defined above.*

A vector space of the form V/W is usually called a *quotient vector space*, or a *quotient space* for short.

Question 7. *Let's understand this in the example in Question 1. Suppose $V = \mathbb{R}^2$, $F = \mathbb{R}$ and $W =$ x-axis, and we want to see whether V/W is a vector space.*

(a) Is the sum of two elements of V/W another element of V/W? (Hint: What is the sum $((x_1, y_1) + W) + ((x_2, y_2) + W)$?)

(b) Is it true that $((x_1, y_1) + W) + ((x_2, y_2) + W) = ((x_2, y_2) + W) + ((x_1, y_1) + W)$ *for all* $(x_1, y_1), (x_2, y_2) \in V$*?*

(c) What would an additive identity be in V/W *under the addition we defined? (Hint: compute* $((x, y) + W) + ((0, 0) + W)$ *for all* $(x, y) \in V$*.)*

(d) If $(x, y) \in V$*, what would the additive inverse of* $((x, y) + W)$ *be in* V/W *under the addition we defined? Represent your answer on a Cartesian plane.*

(e) Is it true that $1 \cdot ((x, y) + W) = ((x, y) + W)$ *for all* $(x, y) \in V$*?*

(f) Is it true that $(c_1 + c_2) \cdot ((x, y) + W) = c_1 \cdot ((x, y) + W) + c_2 \cdot ((x, y) + W)$ *for all* $c_1, c_2 \in F$ *and all* $(x, y) \in V$*?*

There are certainly other axioms that one has to check to prove that V/W *is a vector space in this situation, but I think the above gives you an idea of how the other axioms should be checked.*

The proof of Theorem 2 in full generality will be left as an exercise. Basically, to prove Theorem 2, you will need to check the long list of axioms of a vector space like we did above for the specific example, since V/W is not a subset of any vector space; in particular, one should *always* remember that V/W is NOT a subspace of V.

We will need the following theorem about the dimension of a quotient space.

Theorem 3. *Suppose* V *is a vector space over a field* F *and* W *is a subspace of* V*. If* V *is finite dimensional, then so is* V/W*, and*

$$\dim(V/W) = \dim(V) - \dim(W).$$

Proof. Let $n = \dim(V)$ and $m = \dim(W)$. Then $m \leq n$, and we want to show that $\dim(V/W) = n - m$. Now pick a basis of W. This basis will consists of m vectors; call them $w_1, w_2, \ldots, w_m$. Then extend it to a basis of V; i.e. pick some vectors $v_1, \ldots, v_k \in V$ such that $\{w_1, \ldots, w_m, v_1, \ldots, v_k\}$ is a basis of V. This is always possible by a theorem we proved before the midterm. Now any basis of V has exactly n vectors. So $m+k=n$, i.e. $k=n-m$. We claim that $\{v_1+W,\dots,v_k+W\}$ is a basis of V/W. If that is the case, then $\dim(V/W)=k=n-m$, which is our desired result.

The proof of the claim is left as an exercise. □

Finally, we will prove the following rank-nullity theorem:

Theorem 4 (Rank-nullity theorem)**.** *Suppose V, V' are vector spaces over a field F, and $\dim(V)<\infty$. Suppose $T\colon V\to V'$ is a linear map. Then the image of T is finite dimensional, and*

$$\dim(\mathit{image}(T))=\dim(V)-\dim(\mathit{kernel}(T)).$$

The theorem is such called since dim(image(T)) is sometimes called the *rank* of T, and dim(kernel(T)) is sometimes called the *nullity* of T. The theorem says that the sum of the rank and nullity of a linear map is equal to the dimension of its domain, if the domain is finite dimensional.

The proof consists of several steps. First, let $W=\text{kernel}(T)$, and $U=\text{image}(T)$. We knew from our previous class that W is a subspace of V, and that U is a vector space over F. Since W is a subspace of V, it makes sense to talk about the quotient space V/W. We claim that we can define a new map

$$\tilde{T}\colon V/W\to U$$

such that

$$\tilde{T}(v+W):=T(v)\quad\text{for all } v\in V.$$

There is again a problem of well-definedness; an element of V/W can be written in the form $v+W$ for many different choices of $v\in V$. Hence we need to answer the following question:

Question 8. *If*

$$v_1+W=v_2+W$$

for some $v_1,v_2\in V$, is it true that

$$T(v_1)=T(v_2)?$$

(Hint: The answer had better be yes, and to check that one uses Theorem 1 again, together with the fact that $W=\mathit{kernel}(T)$.)

Now we have checked that the map $\tilde{T}\colon V/W\to U$ we introduced above is well-defined. To proceed further, we claim that $\tilde{T}$ is linear:

Question 9. *Check that* $\tilde{T}\colon V/W \to U$ *is linear. (Hint: What do you need to check? You need to check that*

$$\tilde{T}((v_1 + W) + (v_2 + W)) = \tilde{T}(v_1 + W) + \tilde{T}(v_2 + W)$$

for all $v_1, v_2 \in V$*, and*

$$\tilde{T}(c \cdot (v + W)) = c \cdot \tilde{T}(v + W)$$

for all $c \in F$ *and all* $v \in V$*.)*

Next, we claim that $\tilde{T}\colon V/W \to U$ is an isomorphism between the two vector spaces:

Question 10. *Check that* $\tilde{T}$ *is injective. (Hint: If* $\tilde{T}(v + W) = 0$ *for some* $v \in V$*, what can you say about* v*? Can you conclude that* $v + W = 0 + W$*?)*

Question 11. *Check that* $\tilde{T}$ *is surjective. (Hint: Given* $u \in U$*, there exists* $v \in V$ *such that* $T(v) = u$*. (Why?) Now what is* $\tilde{T}(v + W)$*? Conclude the proof from here.)*

Hence to conclude, $\tilde{T}\colon V/W \to U$ is an isomorphism. Thus

$$\dim(U) = \dim(V/W) = \dim(V) - \dim(W);$$

in other words,

$$\dim(\text{image}(T)) = \dim(V) - \dim(\text{kernel}(T)).$$

This proves the rank-nullity theorem.

Exercises.

1. Let $V = \mathbb{R}^2$ be a vector space over $\mathbb{R}$, and $W = \{(x,y)\colon x+y=0\}$ be a subspace of V. Represent on a Cartesian plane some cosets of W in V. Computer $((0,3)+W)+((0,-4)+W)$ and $3\cdot((0,2)+W)$. Can you describe how you add two elements of V/W with a picture?
2. Let $V = \mathbb{R}^3$ be a vector space over $\mathbb{R}$, and $W_1 = \{(x,y,z)\colon z = x+y\}$ be a subspace of V. Draw some cosets of W_1 in V on a Cartesian coordinate chart. Also, if now $W_2 = \{(x,y,z)\colon x=0, y=z\}$, so that W_2 is also a subspace of V, can you draw some cosets of W_2 in V on a separate picture?
3. Prove Theorem 1. (Hint: Suppose first $v_1 - v_2 \in W$. Then $v_1 = v_2 + w_0$ for some $w_0 \in W$. Now we need to check $v_1 + W \subseteq v_2 + W$, and $v_2 + W \subseteq v_1 + W$. But the former follows, since if $w \in W$, then $v_1 + w = v_2 + w_0 + w \in v_2 + W$. Similarly, one can check that $v_2 + W \subseteq v_1 + W$. Thus $v_1 + W = v_2 + W$, as desired.

   Next, suppose $v_1 + W = v_2 + W$. Then since $v_1 = v_1 + 0$ and $0 \in W$, we have $v_1 \in v_1 + W$. Thus $v_1 \in v_2 + W$, i.e. $v_1 = v_2 + w_0$ for some $w_0 \in W$. Thus $v_1 - v_2 = w_0 \in W$.)
4. Suppose $w_1, \dots, w_m, v_1, \dots, v_k$ are as in the proof of Theorem 3. Let
$$S = \{v_1 + W, \dots, v_k + W\}$$
be a subset of V/W with k elements.
   (a) Prove that S is linearly independent. (Hint: Suppose
$$a_1(v_1 + W) + \cdots + a_k(v_k + W) = 0 + W$$
for some $a_1, \dots, a_k \in F$. Then one has
$$a_1 v_1 + \cdots + a_k v_k \in W$$
(why?), and thus there are coefficients $b_1, \dots, b_m \in F$ such that
$$a_1 v_1 + \cdots + a_k v_k = b_1 w_1 + \cdots + b_m w_m.$$
(why?) Now conclude that $a_1 = \cdots = a_k = 0$.)
   (b) Prove that S spans V/W. (Hint: Pick an element from V/W. Then it is of the form $v+W$ for some $v \in V$. Now there are scalars $a_1, \dots, a_k, b_1, \dots, b_m \in F$ such that
$$v = a_1 v_1 + \cdots + a_k v_k + b_1 w_1 + \cdots + b_m v_m.$$
(why?) Then
$$v + W = a_1(v_1 + W) + \cdots + a_k(v_k + W) + b_1(w_1 + W) + \cdots + b_m(w_m + W).$$
(why?) Now $w_1 + W = \cdots = w_m + W = 0 + W$ (why?). So the above equation says
$$v + W = a_1(v_1 + W) + \cdots + a_k(v_k + W).$$
(why?) This concludes the proof. (why?))

   Together, we proved that S is a basis of V/W, establishing the claim in the proof of Theorem 3.
5. Give a proof of Theorem 2.