Math 403 Spring 2011
Homework 9 Additional problems

1. Let D be the open unit disc centered at 0. Show that if $a \in D$, then
   (i) $\frac{a-z}{1-\bar{a}z} \in D$ for all $z \in D$; and
   (ii) the map $f_a : D \to D$ defined by
   $$f_a(z) = \frac{a-z}{1-\bar{a}z}$$
   is analytic and bijective.
   (Note: The first part shows that $f_a : D \to D$ is well-defined.)
2. Which of the following regions can be mapped bijectively to the open unit disc centered at 0 by an analytic function? Why? (Hint: Explain using the Riemann mapping theorem.)
   (a) $\{z \in \mathbb{C} : z \notin (-\infty, 0]\}$
   (b) $\{z \in \mathbb{C} : 1 < |z| < 2\}$
   (c) $\mathbb{C} \setminus \{1, 2, 3\}$
3. Show that there is no bijective analytic function $f : \mathbb{C} \to D$, where D is the open unit disc centered at 0. (Hint: Use Liouville's theorem.) (Note: This explains why in the statement of the Riemann mapping theorem, one requires the domain to be not all of $\mathbb{C}$.)