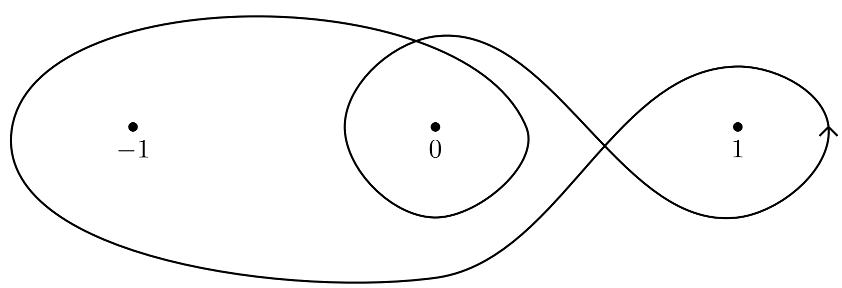
Math 403 Spring 2011
Midterm 2 review

1. Compute the following line integrals:
   (a) $\displaystyle\int_{|z-i|=3} \frac{1}{e^{2\pi z}(z-2i)(z-5i)}dz$
   (b) $\displaystyle\int_{|z|=6} \frac{\sin(\pi z)}{(2z-3)(z-2)}dz$
   (c) $\displaystyle\int_{|z|=1} \frac{e^z}{z^{2011}}dz$
   (d) $\displaystyle\int_{|z|=1} z^{2011}e^{1/z}dz$
   (e) $\displaystyle\int_{|z|=1} \frac{z}{\sin^2 z}dz$
   (f) $\displaystyle\int_{|z|=1} \frac{1}{z(1-\cos z)}dz$
   (g) $\displaystyle\int_{|z-1|=5} \frac{e^z}{z^2(z-3)}dz$
   (h) $\displaystyle\int_{|z-\pi|=1} \frac{1}{\sin z}dz$

   The circles should be oriented counter-clockwise.
2. Suppose γ is the following curve:



   Compute $\displaystyle\int_\gamma \frac{e^{2z}}{z^2(1-z^2)}dz$.
3. Find the principal parts of the Laurent series of the following functions when they are expanded in a small punctured disc centered at 0. Also, determine whether 0 is a removable singularity, pole or essential singularity of the function in each case. If it is a pole, find also the order of the pole. Find also the residue of the functions at 0.
   (a) $\displaystyle\frac{z+1}{z^3(z+2i)}$
   (b) $\displaystyle\frac{1}{z^3(z+1)e^z}$ (Hint: Better write $1/e^z$ as e^{-z} instead of carrying out a division of power series!)

(c) $\dfrac{1}{z^2(z+1)\sin z}$

(d) $z^3\cos(1/z)$

(e) $\dfrac{1}{z^2-z}-\dfrac{1}{z}$

4. Expand the function $\dfrac{\sin(2z)}{(z-3\pi)^6}$ in Laurent series in a small punctured disc centered at 3π. Is 3π a removable singularity, pole or essential singularity of the function? If it is a pole, what is its order? Find also the residue of the function at 3π.
5. Compute the following integrals using complex analysis. You should give a careful argument when you need to estimate certain integrals. (On the other hand, a useful fact to know is that if you have a rational function, i.e. the quotient of two polynomials $P(z)/Q(z)$, then the maximum of $P(z)/Q(z)$ over the circle of radius R centered at the origin is comparable to, up to a multiplicative constant, $1/R^m$ as $R\to\infty$, where m is the degree of Q minus the degree of P. You will be allowed to use this (only correctly!) in the midterm.)

   (a) $\displaystyle\int_0^{2\pi}\frac{dx}{5+4\cos x}$ (Answer: $\frac{\pi}{3}$)

   (b) $\displaystyle\int_{-\infty}^{\infty}\frac{1}{(x^2+1)^2}dx$ (Answer: $\frac{\pi}{2}$)

   (c) $\displaystyle\int_{-\infty}^{\infty}\frac{\cos(3x)}{(x^2+1)^2dx}$ (Answer: $\frac{2\pi}{e^3}$)

   (d) $\displaystyle\int_{-\infty}^{\infty}\frac{\sin x}{x^2+4x+5}dx$ (Answer: $-\frac{\pi}{e}\sin 2$)

   (e) $\displaystyle\int_0^{\infty}\frac{1-\cos x}{x^2}dx$ (Hint: Use the indented semi-circle, as in Figure 2.16 of your textbook. In fact, this integral is just a variant of the one in Example 6 of Section 2.6 of your text.) (Answer: $\frac{\pi}{2}$)
6. Find the number of zeroes of the polynomial $P(z)=z^{10}-z^4+7iz^3+2z^2+z-1$ in each of the following regions (counting multiplicities):

   (a) the closed unit disc $\{z\colon |z|\le 1\}$

   (b) the closed annulus $\{z\colon 1\le |z|\le 2\}$

   (c) the closed annulus $\{z\colon |z|\ge 2\}$.
7. Find the number of solutions of the equation $4z^3=e^z$ in the closed unit disc $\{z\colon |z|\le 1\}$, counting multiplicities.
8. Find the number of zeroes of $5ze^z-1$ in the closed unit disc $\{z\colon |z|\le 2\}$, counting multiplicities.
9. Find the number of solutions of the equation $z+3=e^z$ on the half plane $\{z\colon \text{Re } z\le 0\}$, counting multiplicities.
10. Suppose $f(z)$ is analytic in a small disc centered at z_0, and z_0 is a zero of $f(z)$ of order 1. Show that

$$\int_\gamma \frac{1}{f(z)}dz = 2\pi i\frac{1}{f'(z_0)}$$

   if γ is a sufficiently small circle centered at z_0. This is a very useful fact to know when computing line integrals. You should try using this to solve Question 1(h) above.

11. Suppose $g(z)$ is analytic in a small disc centered at z_0. Show that the residue of the function $\frac{g(z)}{(z-z_0)^k}$ is equal to

$$\frac{1}{(k-1)!}g^{(k-1)}(z_0)$$

if k is a positive integer. Again, this gives you a very easy way of computing the line integral of $\frac{g(z)}{(z-z_0)^k}$ around a small circle centered at z_0 (how?). You should try using this to solve Questions 1(c)(g), Question 2 and compute the residues in Questions 3(b)(c) again.

12. Suppose $h(z)$ is analytic in a small disc centered at z_0, and $h(z_0) \neq 0$. Show that

$$\int_\gamma \frac{1}{(z-z_0)^2 h(z)} dz = -2\pi i \frac{h'(z_0)}{h^2(z_0)}$$

if γ is a sufficiently small circle centered at z_0. (Hint: Use the previous question.)

13. The purpose of this question is to show the remarkable identity

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^2}, \tag{1}$$

which holds whenever z is a complex number that is not an integer. (In particular, this is true if z is a real number but not an integer, and this is already not so easy to prove without complex analysis[1]!) There are two derivations, both using complex analysis, and we will carry out both.

(a) Fix z in $\mathbb{C} \setminus \mathbb{Z}$. Consider the function

$$f(w) = \frac{\pi \cot(\pi w)}{(w+z)^2}.$$

(i) Show that every integer is a pole of $f(w)$, and that the only other pole of $f(w)$ is at $w = -z$.

(ii) Suppose n is an integer. Show that the residue of $f(w)$ at $w = n$ is $\frac{1}{(z+n)^2}$.

(iii) Show that the residue of $f(w)$ at $-z$ is $-\frac{\pi^2}{\sin^2(\pi z)}$.

(iv) Show that there is some constant C such that

$$\max_{|z|=N+\frac{1}{2}} |f(z)| \leq \frac{C}{N^2}$$

for all positive integer N.

(v) Show that

$$\lim_{N\to\infty} \int_{|z|=N+\frac{1}{2}} f(z)dz \to 0,$$

where N tends to infinity along the positive integers.

(vi) Conclude that (1) holds.

[1] This can be proved, on the other hand, using the Poisson summation formula in Fourier analysis if z is real and non-integral.

(b) Consider the function

$$g(z) = \frac{\pi^2}{\sin^2(\pi z)} - \sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^2}.$$

This function is analytic in z except possibly at the integers. We want to show that $g(z)$ is identically zero.

(i) Compute the principal part of the Laurent series expansion of the function

$$\frac{\pi^2}{\sin^2(\pi z)}$$

in a small punctured disc centered at $z = 0$. Hence, conclude that 0 is a removable singularity of $g(z)$.

(ii) Show that $g(z + m) = g(z)$ for any integer m.

(iii) Show that any integer is a removable singularity of $g(z)$. (Hint: Use (i) and (ii) above.) It follows that $g(z)$ extends to an entire function on $\mathbb{C}$.

(iv) Show that

$$\max_{|\mathrm{Im}\ z| = y} |g(z)| \to 0 \quad \text{as } y \to +\infty.$$

(v) Show that $g(z)$ is a bounded function. (Hint: Use (ii) and (iv).)

(vi) Conclude that $g(z)$ is constant. (Hint: Liouville's theorem.)

(vii) Conclude that $g(z)$ is identically zero. (Hint: Use (iv) and (vi).)