MATH6081A Homework 1

1. Prove Young's convolution inequality. (Hint: If $f \in L^p$ and $g \in L^q$ are non-negative, without loss of generality assume $\|f\|_p = \|g\|_q = 1$. Then for $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, we have

$$f * g(x) \leq \int f(x-y)^{\frac{p}{r}} g(y)^{\frac{q}{r}} f(x-y)^{\frac{r-p}{r}} g(y)^{\frac{r-q}{r}} dy.$$

Applying Hölder's inequality with

$$1 = \frac{1}{r} + \left(\frac{1}{p} - \frac{1}{r}\right) + \left(\frac{1}{q} - \frac{1}{r}\right),$$

and using the normalizations of f and g, we get

$$f * g(x) \leq \left(\int f(x-y)^p g(y)^q dy\right)^{\frac{1}{r}}.$$

Raising both sides to power r and integrate, we obtain the desired inequality. Alternatively, we can rewrite this proof more symmetrically using duality: let $s = r'$ so that $\frac{1}{p} + \frac{1}{q} + \frac{1}{s} = 2$. Then

$$\iint f(x-y)g(y)h(x)dydx = \iint [f(x-y)^p g(y)^q]^{1-\frac{1}{s}} [f(x-y)^p h(x)^s]^{1-\frac{1}{q}} [g(y)^q h(x)^s]^{1-\frac{1}{p}} dydx.$$

Apply Hölder's inequality with

$$1 = \left(1 - \frac{1}{s}\right) + \left(1 - \frac{1}{q}\right) + \left(1 - \frac{1}{p}\right),$$

we obtain the desired inequality.)

2. For $x \in \mathbb{T}$, let

$$D_N(x) = \sum_{|n| \leq N} e^{2\pi i n x} \quad \text{for } N \in \mathbb{N} \cup \{0\}$$

and

$$F_N(x) = \frac{D_0(x) + D_1(x) + \cdots + D_{N-1}(x)}{N} \quad \text{for } N \in \mathbb{N}.$$

Find a closed formula for both $D_N(x)$ and $F_N(x)$.

3. (a) Show that if $f \in L^p(\mathbb{T})$, $1 \leq p < \infty$, then

$$\lim_{N \to \infty} \|f * F_N - f\|_{L^p(\mathbb{T})} = 0.$$

(Hint: Approximate f by a continuous function.)

(b) Show that if $f \in L^1(\mathbb{T})$ and

$$\widehat{f}(n) = 0$$

for all $n \in \mathbb{Z}$, then $f(x) = 0$ a.e. Hence if $f, g \in L^1(\mathbb{T})$ and

$$\widehat{f}(n) = \widehat{g}(n)$$

for all $n \in \mathbb{Z}$, then $f(x) = g(x)$ a.e. (Hint: Use (a).)

4. Let $f\colon \mathbb{T} \to \mathbb{C}$. Show that

   (a) If $f \in C^k(\mathbb{T})$ for some positive integer k, then

   $$|\widehat{f}(n)| \lesssim (1+|n|)^{-k}.$$

   In particular the Fourier series of f converges absolutely and uniformly if $k \geq 2$.

   (b) If f is Lipschitz on $\mathbb{T}$, then

   $$|\widehat{f}(n)| \lesssim (1+|n|)^{-1}.$$

   (c) If f is Hölder continuous of some order $\alpha \in (0,1)$ on $\mathbb{T}$, then

   $$|\widehat{f}(n)| \lesssim (1+|n|)^{-\alpha}.$$

   Show also that

   $$\sum_{2^k \leq |n| < 2^{k+1}} |\widehat{f}(n)|^2 \lesssim 2^{-2k\alpha}$$

   for all positive integers k, so the Fourier series of f converges absolutely and uniformly as long as $\alpha > 1/2$. (Hint: For the second estimate, consider the L^2 norm of $F_h(x) := f(x+h) - f(x)$ for some h with $|h| \simeq 2^{-k}$.)

5. Prove the Riemann-Lebesgue lemma for the Fourier transform on $\mathbb{R}^n$: if $f \in L^1(\mathbb{R}^n)$, then $\widehat{f}$ is continuous on $\mathbb{R}^n$ and vanishes at infinity. (Hint: Approximate $f \in L^1(\mathbb{R}^n)$ by a Schwartz function on $\mathbb{R}^n$.) How do you prove a similar statement on $\mathbb{T}$?

6. (a) Show that if $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, then

   $$\lim_{y \to 0} \|\tau_y f - f\|_{L^p(\mathbb{R}^n)} = 0,$$

   where τ_y is the translation given by $\tau_y f(x) := f(x+y)$ for $y \in \mathbb{R}^n$.

   (b) What can you say if $p = \infty$?

7. Compute the Fourier transform of the following L^1 functions of x on $\mathbb{R}^n$:

   (a) $e^{-\pi|x|^2}$

   (b) $e^{-\pi t|x|^2}$, where $t > 0$

   (c) $\dfrac{1}{(1+|x|^2)^{\frac{n+1}{2}}}$

   (Hint: Write

   $$\frac{1}{(1+|x|^2)^{\frac{n+1}{2}}} = \frac{\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} \int_0^\infty e^{-\pi t(1+|x|^2)} t^{\frac{n+1}{2}} \frac{dt}{t}$$

   and use (b). This reduces the problem to the computation of an integral independent of n, and to compute that integral amounts to computing the Fourier transform of $\frac{1}{1+x^2}$ on $\mathbb{R}$, a task that can be accomplished by, for instance, calculus of residues. See also the *method of subordination*.)

(d) $\dfrac{y}{(y^2+|x|^2)^{\frac{n+1}{2}}}$, where $y > 0$

8. If A is a positive definite real symmetric $n \times n$ matrix, show that the Fourier transform of $e^{-\pi\langle x, Ax\rangle}$ on $\mathbb{R}^n$ is $(\det A)^{-1/2} e^{-\pi\langle x, A^{-1}x\rangle}$. This generalizes Question 7(b).

9. From the answer of Question 7(d), it is easy to evaluate the Fourier transform of the L^1 function $e^{-2\pi y|x|}$ of $x \in \mathbb{R}^n$ for all $y > 0$ (how?). In this question we give an alternative direct evaluation of the same Fourier transform, that does not require knowing the answer ahead of time.

   (a) Compute directly the Fourier transform of $e^{-2\pi|x|}$ in 1-dimension by evaluating
   $$\int_{\mathbb{R}} e^{-2\pi|x|} e^{-2\pi i x\xi} dx.$$
   Hence deduce that
   $$e^{-2\pi|u|} = \int_{\mathbb{R}} \frac{e^{2\pi i u v}}{\pi(1+v^2)} dv$$
   for all $u \in \mathbb{R}$.

   (b) Using the identity
   $$\frac{1}{\pi(1+v^2)} = \int_0^\infty e^{-\pi t(1+v^2)} dt$$
   (which is just a version of the display equation in the hint of Question 7(c)), express $e^{-2\pi|u|}$ as a suitable weighted average of $e^{-\pi t u^2}$ over $t \in (0,\infty)$ for all $u \in \mathbb{R}$.

   (c) The above allows us to write $e^{-2\pi|x|}$ as a suitable weighted average of $e^{-\pi t|x|^2}$ over $t \in (0,\infty)$ for every $x \in \mathbb{R}^n$. Combine this with the answer of Question 7(b) to evaluate the Fourier transform of $e^{-2\pi|x|}$ on $\mathbb{R}^n$.

   (d) Hence evaluate the Fourier transform of $e^{-2\pi y|x|}$ on $\mathbb{R}^n$ for every $y > 0$.

10. Compute the k-fold convolution $e^{-\pi a_1|x|^2} * e^{-\pi a_2|x|^2} * \cdots * e^{-\pi a_k|x|^2}$ on $\mathbb{R}^n$, if $a_1, \ldots, a_k$ are positive real numbers. (Hint: What is the fastest way here?)

11. Prove the following inequality of Hardy's: If $p \in (1,\infty)$, then for any non-negative measurable function f on $(0,\infty)$, we have
    $$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)^p dx.$$
    (Hint: Write $\frac{1}{x}\int_0^x f(t)dt$ as $\int_0^1 f(xt)dt$ and use Minkowski's inequality. Alternatively, let $F(x) = x^{1/p} f(x)$. Then the inequality to be proved can be reformulated as
    $$\left\| \int_0^\infty F(xt) g(t^{-1}) \frac{dt}{t} \right\|_{L^p(\mathbb{R}_+, \frac{dx}{x})} \leq \frac{p}{p-1} \|F(x)\|_{L^p(\mathbb{R}_+, \frac{dx}{x})}$$
    where $g(t) := t^{\frac{1}{p}-1}\chi_{[1,\infty)}(t)$. This is a convolution inequality on the multiplicative group $\mathbb{R}_+$, where $\frac{dx}{x}$ is the Haar measure on the group. Applying Young's convolution inequality yields the desired estimate.)

12. For $p \in (1, \infty)$, let
$$C_p := \int_0^\infty \frac{dt}{t^{1/p}(1+t)} < \infty.$$

   (a) Show that
$$C_p = \frac{\pi}{\sin(\pi/p)}.$$
   (Hint: Use contour integration.)

   (b) For $f \colon \mathbb{R}_+ \to \mathbb{C}$ (say bounded with compact support) and $x \in \mathbb{R}_+$, we define the Hilbert integral of f by
$$Hf(x) = \int_0^\infty \frac{f(t)}{x+t} dt.$$
   Show that for $p \in (1, \infty)$, we have
$$\left(\int_0^\infty |Hf(x)|^p dx \right)^{1/p} \leq C_p \left(\int_0^\infty |f(x)|^p dx \right)^{1/p}.$$
   Also show that this inequality is false if we replace C_p by anything strictly smaller. (Hint: To prove the positive result, a similar strategy to the previous question works. You can rewrite $Hf(x)$ as $\int_0^\infty \frac{f(xt)}{1+t} dt$ and use Minkowski's inequality. Alternatively, let $F(x) = x^{1/p} f(x)$. Then the inequality to be proved can be reformulated as
$$\left\| \int_0^\infty F(xt) g(t^{-1}) \frac{dt}{t} \right\|_{L^p(\mathbb{R}_+, \frac{dx}{x})} \leq C_p \|F(x)\|_{L^p(\mathbb{R}_+, \frac{dx}{x})}$$
   where $g(t) := \frac{t^{1/p}}{1+t}$. This is a convolution inequality on the multiplicative group $\mathbb{R}_+$, where $\frac{dx}{x}$ is the Haar measure on the group. Applying Young's convolution inequality yields the desired estimate. Now examine the proof of the positive result, to show that the constant C_p cannot be improved.)

13. We record here three equivalent reformulations of Hölder's inequality that are often useful, and give one application.

   (a) If $p, q, r \in (0, \infty]$ and $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$, then for any measurable functions f and g,
$$\|fg\|_{L^p} \leq \|f\|_{L^q} \|g\|_{L^r}.$$

   (b) If $\theta \in [0, 1]$ and $p \in (0, \infty]$, then for any non-negative measurable functions F and G,
$$\|F^{1-\theta} G^\theta\|_{L^p} \leq \|F\|_{L^p}^{1-\theta} \|G\|_{L^p}^\theta.$$

   (c) If $\theta \in [0, 1]$ and $p, p_0, p_1 \in (0, \infty]$ with $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, then for any non-negative measurable functions F and G,
$$\|F^{1-\theta} G^\theta\|_{L^p} \leq \|F\|_{L^{p_0}}^{1-\theta} \|G\|_{L^{p_1}}^\theta.$$

(d) In particular, if $\theta \in [0,1]$ and $p, p_0, p_1 \in (0,\infty]$ with $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, then for any measurable function f,

$$\|f\|_{L^p} \leq \|f\|_{L^{p_0}}^{1-\theta} \|f\|_{L^{p_1}}^{\theta}.$$

Indeed, (a) is clearly equivalent to the more familiar form of Hölder's inequality (which is the case $p = 1$ of (a)). Applying (a) with $f = F^{1-\theta}$, $g = G^{\theta}$, $q = p_0/(1-\theta)$, $r = p_1/\theta$, we obtain (c). Specializing in (c) to the case $p_0 = p_1 = p$, we get (b). Applying (b) with $\theta = p/r$, $1-\theta = p/q$, $F = |f|^{1/(1-\theta)}$ and $G = |g|^{1/\theta}$, we obtain (a). (d) follows from (c) by setting $F = G = |f|$. The inequality in (d) is a baby case of interpolation, which we will take up in Lecture 8.