MATH6081A Homework 2

1. (a) Show that if u is a locally integrable function on $\mathbb{R}^n$ that grows at most polynomially at infinity, then

$$u(f) := \int_{\mathbb{R}^n} u(x)f(x)dx$$

defines a tempered distribution on $\mathbb{R}^n$.

   (b) Show that $\delta_x(f) := f(x)$ defines a tempered distribution on $\mathbb{R}^n$ for any $x \in \mathbb{R}^n$, that is homogeneous of degree $-n$.

   (c) Show that the linear functional

$$f \mapsto \lim_{\varepsilon \to 0^+} \int_{|x|>\varepsilon} f(x)\frac{dx}{x}$$

on $\mathcal{S}(\mathbb{R})$ defines a tempered distribution on $\mathbb{R}$, that is homogeneous of degree -1. More generally, on $\mathbb{R}^n$, show that for any $1 \leq j \leq n$, the linear functional

$$f \mapsto \lim_{\varepsilon \to 0^+} \int_{|x|>\varepsilon} f(x)\frac{x_j}{|x|^{n+1}}dx$$

on $\mathcal{S}(\mathbb{R}^n)$ defines a homogeneous tempered distribution on $\mathbb{R}^n$ of degree $-n$. (These are usually referred to as the principal value of $\frac{1}{x}$ and $\frac{x_j}{|x|^{n+1}}$ respectively.)

2. Show that the following linear maps on $\mathcal{S}(\mathbb{R}^n)$ are continuous. Also explain how the continuity of these maps on $\mathcal{S}(\mathbb{R}^n)$ allows one to extend these maps from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$. Finally, prove that the extended maps are still continuous on $\mathcal{S}'(\mathbb{R}^n)$.

   (a) Translations: $f(x) \mapsto f(x+h)$, for any fixed $h \in \mathbb{R}^n$

   (b) Differentiations: $f(x) \mapsto D^\alpha f(x)$, for any fixed multiindex α

   (c) Multiplication by polynomials: $f(x) \mapsto P(x)f(x)$ for any fixed polynomial $P(x)$

   (d) The Fourier transform: $f(x) \mapsto \widehat{f}(x)$

3. (a) Show that the Fourier transform defines a bijection on the space of tempered distributions on $\mathbb{R}^n$.

   (b) Show that the Fourier transform of a homogeneous tempered distribution of degree α on $\mathbb{R}^n$ is a homogeneous tempered distribution of degree $-n-\alpha$.

4. In what sense does the distributional derivative of a C^1 function on $\mathbb{R}^n$ agree with the classical derivative?

5. Suppose $0 < \alpha < n$.

   (a) Show that

$$\frac{\pi^{\alpha/2}}{\Gamma(\alpha/2)} \int_\varepsilon^R e^{-\pi t|x|^2} t^{\alpha/2} \frac{dt}{t} \to |x|^{-\alpha}$$

in the topology of $\mathcal{S}'(\mathbb{R}^n)$ as $\varepsilon \to 0$ and $R \to +\infty$. (Hint: Note that

$$\left| \int_0^\varepsilon e^{-\pi t|x|^2} t^{\alpha/2} \frac{dt}{t} \right| \lesssim_\alpha \varepsilon^{\alpha/2}$$

and

$$\left| \int_R^\infty e^{-\pi t|x|^2} t^{\alpha/2} \frac{dt}{t} \right| \lesssim_\beta \int_R^\infty (t|x|^2)^{-\beta/2} t^{\alpha/2} \frac{dt}{t} \lesssim_{\alpha,\beta} R^{(\alpha-\beta)/2}|x|^{-\beta}$$

for any $\beta \in (\alpha, n)$.)

(b) Hence, or otherwise, compute the Fourier transform of the tempered distribution on $\mathbb{R}^n$ given by $|x|^{-\alpha}$. (Hint: Use the Fourier transform of $e^{-\pi t|x|^2}$ from Homework 1.)

6. (a) Show that

$$\langle u_n, f \rangle := \int_{|\xi|\geq 1} \frac{f(\xi)}{|\xi|^n} d\xi + \int_{|\xi|\leq 1} \frac{f(\xi) - f(0)}{|\xi|^n} d\xi$$

defines a tempered distribution u_n on $\mathbb{R}^n$, that agrees with $|\xi|^{-n}$ away from the origin. In addition, if n is even (so that $|\xi|^n$ is a polynomial in ξ), then $|\xi|^n u_n = 1$ as elements of $\mathcal{S}'(\mathbb{R}^n)$.

(b) For $0 < \alpha < n$, let u_α be the tempered distribution on $\mathbb{R}^n$ given by

$$\langle u_\alpha, f \rangle := \int_{|\xi|\geq 1} \frac{f(\xi)}{|\xi|^\alpha} d\xi + \int_{|\xi|\leq 1} \frac{f(\xi) - f(0)}{|\xi|^\alpha} d\xi.$$

Show that $u_\alpha \to u_n$ in the topology of $\mathcal{S}'(\mathbb{R}^n)$ as $\alpha \to n^-$.

(c) Hence compute the inverse Fourier transform of u_n. (Hint: Use Question 5(b).)

7. Compute the inverse Fourier transform of the tempered distribution $\operatorname{sgn}(\xi)$ on $\mathbb{R}$. (Hint: $\operatorname{sgn}(\xi)\chi_{[-n,n]}(\xi)$ converges in $\mathcal{S}'(\mathbb{R})$ to $\operatorname{sgn}(\xi)$ as $n \to \infty$. Also

$$\int_{-n}^n \operatorname{sgn}(\xi) e^{2\pi i x\xi} d\xi = 2i \int_0^n \sin(2\pi x\xi) d\xi = \frac{i}{\pi x}(1 - \cos(2\pi n x)).$$

We now need to evaluate the limit of this in $\mathcal{S}'(\mathbb{R})$ as $n \to \infty$; in other words, we need to pair it this with a Schwartz function $f(x)$ and let $n \to \infty$. But then we may replace $f(x)$ by $[f(x) - f(-x)]/2$ since the kernel is odd, and we note that

$$\int_{\mathbb{R}} \cos(2\pi n x) \frac{f(x) - f(-x)}{2x} dx \to 0$$

as $n \to \infty$ by the Riemann-Lebesgue lemma. Thus in the limit we get

$$\int_{\mathbb{R}} \frac{i}{2\pi x}[f(x) - f(-x)] dx = \lim_{\varepsilon \to 0^+} \int_{|x|\geq\varepsilon} \frac{i f(x)}{\pi x} dx,$$

which shows that the inverse Fourier transform of $\operatorname{sgn}(\xi)$ is p.v.$\frac{i}{\pi x}$. Alternatively, write $\operatorname{sgn}(\xi)$ as ξu_1 where u_1 is the distribution on $\mathbb{R}$ defined in Question 6(a). Thus the inverse Fourier transform of $\operatorname{sgn}(\xi)$ is the derivative of the inverse Fourier transform of u_1 divided by $2\pi i$.)

8. Compute the inverse Fourier transform of the tempered distribution $\frac{\xi_j}{|\xi|}$ on $\mathbb{R}^n$ for any $1 \leq j \leq n$. (Hint: Modify the second approach in the hint to Question 7.)

9. Suppose $n \in \mathbb{N}$. For $1 \leq j \leq n$, let

$$f_j(\xi) = \frac{\xi_j}{|\xi|} e^{-2\pi|\xi|}, \quad \xi \in \mathbb{R}^n.$$

Compute the inverse Fourier transform of these L^1 functions on $\mathbb{R}^n$. Hence compute the inverse Fourier transform of $\frac{\xi_j}{|\xi|} e^{-2\pi y|\xi|}$ for any $y > 0$.
(Hint: $f_j(\xi) = \partial_j e^{-2\pi|\xi|}$ in the sense of distributions. Now use the inverse Fourier transform of $e^{-2\pi|\xi|}$ from Question 9 of Homework 1, and the fact that the distributional Fourier transform agrees with the L^1 Fourier transform when applied to an L^1 function.)

10. Show that if $u \in \mathcal{S}'(\mathbb{R}^n)$ and $g \in \mathcal{S}(\mathbb{R}^n)$, then $u * g$ is a C^∞ function, given by

$$u * g(x) = \langle u, \tau_x \tilde{g} \rangle$$

where $\tilde{g}(y) := g(-y)$ and $[\tau_x \tilde{g}](y) := g(x - y)$. Also verify that

$$\widehat{u * g} = \widehat{u} \cdot \widehat{g}.$$

11. Fix a function $\zeta \in C_c^\infty(\mathbb{R})$ with $\zeta(y) = 1$ for $|y| \leq 1$ and define, for each $s \in \mathbb{C}$ with $\operatorname{Re} s > 0$, a locally integrable function on $\mathbb{R}$ by

$$\alpha_s(y) = \frac{1}{\Gamma(s)} \zeta(y) y^{s-1} \chi_{[0,\infty)}(y).$$

for all $f \in \mathcal{S}(\mathbb{R})$.

(a) Show that for any such s and f, we have

$$\langle \alpha_s, f \rangle = \frac{(-1)^N}{\Gamma(s)s(s+1)\dots(s+N-1)} \int_0^\infty \left(\frac{d}{dy}\right)^N [f(y)\zeta(y)] y^{N+s} \frac{dy}{y}$$

for all $N \in \mathbb{N}$.

(b) Also show that for every $f \in \mathcal{S}(\mathbb{R})$, the right hand side actually defines a holomorphic function of s on the half plane $\{\operatorname{Re} s > -N\}$.

(c) The above allows us to define the analytic continuation of α_s to the right half plane $\{\operatorname{Re} s > -N\}$ for any $N \in \mathbb{N}$. With this extension, verify that

$$\alpha_0 = \delta_0,$$

the delta function at 0, and more generally that α_{-k} is the k-th derivative of the delta function at 0 for all $k \in \mathbb{N}$.

12. Solve the Cauchy problem of the heat equation on $\mathbb{R}^n$

$$\begin{cases} \partial_t u = \Delta u & \text{on } [0, \infty) \times \mathbb{R}^n \\ u(x, 0) = f(x) & \text{for } x \in \mathbb{R}^n \end{cases}$$

by taking the Fourier transform in x if $f \in \mathcal{S}(\mathbb{R}^n)$. Perform a similar calculation for the Schrodinger and the wave equations on $\mathbb{R}^n$.

13. Suppose $n \geq 1$, and that u is a bounded harmonic function on $\mathbb{R}^{n+1}_+$. If u extends to a continuous function on the closure of the upper half space $\mathbb{R}^{n+1}_+$ and the extended u vanishes on the boundary of $\mathbb{R}^{n+1}_+$, show that u is identically zero.

    (Hint: Reflect u across the boundary of $\mathbb{R}^{n+1}_+$, and use Liouville's theorem: every bounded harmonic function on $\mathbb{R}^{n+1}$ is constant.)

    This shows that if $f \in \mathcal{S}(\mathbb{R}^n)$ and $u(x,y)$ is the Poisson integral of f, then u is the *unique* harmonic extension of f on $\mathbb{R}^{n+1}_+$ that is bounded on $\mathbb{R}^{n+1}_+$ and continuous up to the boundary.

14. Let $n \geq 1$ and $\mathbb{R}^{n+1}_+ = \{(x_0, \dots, x_n) \colon x_0 > 0\}$. The following is an alternative way of arriving at the Poisson integral formula on $\mathbb{R}^{n+1}_+$ other than the one given in the lecture.

    (a) Show that if y^* is the reflection of y across the x_0 axis, i.e. if $y^* = (-y_0, y')$ whenever $y = (y_0, y')$, then

    $$G(x,y) = -\frac{1}{2\pi}(\log|x-y| - \log|x-y^*|)$$

    is the Green's function of $-\Delta$ on $\mathbb{R}^2_+$, i.e.

    $$v(x) = \int_{\mathbb{R}^2_+} G(x,y) w(y) dy$$

    solves

    $$\begin{cases} -\Delta v(x) = w(x) & \text{for all } x \in \mathbb{R}^2_+ \\ v(0,x') = 0 & \text{for all } x' \in \mathbb{R} \end{cases}$$

    whenever $w \in C^\infty_c(\mathbb{R}^2_+)$. Similarly, show that if $n \geq 2$, then

    $$G(x,y) = \frac{\Gamma((n-2)/2)}{4\pi^{n/2}} \left(\frac{1}{|x-y|^{n-2}} - \frac{1}{|x-y^*|^{n-2}} \right)$$

    is the Green's function of $-\Delta$ on $\mathbb{R}^{n+1}_+$.

    (b) Verify the Gauss-Green formula on $\mathbb{R}^{n+1}_+$: if u, v are smooth up to the boundary of $\mathbb{R}^{n+1}_+$,

    $$\sup_{x \in \mathbb{R}^{n+1}_+} (|u(x)| + |\partial_x u(x)|) < +\infty$$

    and

    $$\lim_{R\to\infty} \sup_{\substack{x \in \mathbb{R}^{n+1}_+ \\ |x| \geq R}} (|v(x)| + |\partial_x v(x)|) = 0,$$

    then

    $$-\int_{\mathbb{R}^{n+1}_+} (u\Delta v - v\Delta u) dx = \int_{\mathbb{R}^n} (u\partial_{x_0} v - v\partial_{x_0} u) dx'.$$

(c) Let $f \in \mathcal{S}(\mathbb{R}^n)$. Let u be a solution to the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{on } \mathbb{R}^{n+1}_+ \\ u(0,x') = f(x') & \text{for all } x' \in \mathbb{R}^n \end{cases}$$

where u is smooth up to the boundary of $\mathbb{R}^{n+1}_+$, and u, $\partial_x u$ are both bounded on $\mathbb{R}^{n+1}_+$. Show that for any $y \in \mathbb{R}^{n+1}_+$, we have

$$u(y) = \int_{\mathbb{R}^n} f(x') \left. \frac{\partial}{\partial x_0} G(x,y) \right|_{x_0=0} dx'.$$

Verify also that

$$\left. \frac{\partial}{\partial x_0} G(x,y) \right|_{x_0=0} = P_{y_0}(y'-x')$$

where $P_{y_0}(y')$ is the Poisson kernel in $\mathbb{R}^{n+1}_+$. (Hint: Fix $y \in \mathbb{R}^{n+1}_+$. Let w be a smooth function with compact support on $\mathbb{R}^{n+1}$ with $\int_{\mathbb{R}^{n+1}} w = 1$. Let $w_\varepsilon(x) = \varepsilon^{-(n+1)} w(\varepsilon^{-1}(x-y))$, and let

$$v_\varepsilon(x) = \int_{\mathbb{R}^{n+1}_+} G(x,z) w_\varepsilon(z) dz$$

for all sufficiently small $\varepsilon > 0$. Apply the Gauss-Green formula to u and v_ε, and let ε tend to 0.)

15. Let $n \geq 2$, and $\mathbb{B}^n$ be the unit ball on $\mathbb{R}^n$ centered at the origin. Let $\partial\mathbb{B}^n$ be the boundary of $\mathbb{B}^n$ (i.e. the unit sphere in $\mathbb{R}^n$), and $d\sigma(y)$ be the surface measure on $\partial\mathbb{B}^n$ induced from the Lebesgue measure on $\mathbb{R}^n$. The Poisson integral formula for $\mathbb{B}^n$ says that if f is continuous on $\partial\mathbb{B}^n$, then

$$u(x) = \int_{\partial\mathbb{B}^n} P(x,y) f(y) d\sigma(y),$$

where

$$P(x,y) = \frac{\Gamma(n/2)}{2\pi^{n/2}} \frac{1-|x|^2}{|x-y|^n}.$$

One can arrive at this formula from the corresponding formula on the upper half space $\mathbb{R}^n_+$ via the stereographic projection (which maps $\mathbb{B}^n$ diffeomorphically onto $\mathbb{R}^n_+$ and preserves harmonic functions; indeed the stereographic projection is an isometry from $\mathbb{B}^n$ to $\mathbb{R}^n_+$ if we put the hyperbolic metric on both spaces, and Euclidean harmonic functions on the two spaces agree with the hyperbolic harmonic functions on the two spaces). Alternatively, one can mimic what we do in Question 13, and give a direct derivation of the Poisson integral formula on $\mathbb{B}^n$ by first deriving the Green's function for $\mathbb{B}^n$. In this question, we will give yet another derivation, which is simple and based on the mean-value property of harmonic functions. (In 2-dimensions, one can also use complex analysis, via the Cauchy integral formula, but we will not enter into this here.) From now on suppose u is harmonic on $\mathbb{B}^n$ and continuous up to the boundary of $\mathbb{B}^n$.

(a) Show that

$$u(0) = \frac{1}{\omega_{n-1}} \int_{\partial \mathbb{B}^n} u(y) d\sigma(y)$$

where $\omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the surface area of $\partial \mathbb{B}^n$ with respect to $d\sigma$. (Hint: Consider the average of u on the sphere of radius r centered at the origin. Differentiate with respect to r to show that this is constant in r.)

(b) Fix $x_0 \in \mathbb{B}^n$. Let

$$v(x) = u\left(\frac{(1 + 2x_0 \cdot x + |x|^2)x_0 + (1 - |x_0|^2)x}{1 + 2x_0 \cdot x + |x_0|^2|x|^2} \right).$$

Show that v is also harmonic on harmonic on $\mathbb{B}^n$ and continuous up to the boundary of $\mathbb{B}^n$. Apply part (a) to v instead of u and conclude that

$$u(x_0) = \int_{\partial \mathbb{B}^n} P(x_0, y) u(y) d\sigma(y).$$