MATH6081A Homework 3

1. Show that if f is a measurable function on $\mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} |f(x)|^p dx = \int_0^\infty p\alpha^{p-1} |\{x \in \mathbb{R}^n \colon |f(x)| > \alpha\}| d\alpha$$

for any $0 < p < \infty$. (Hint: Interpret the measure in the integral on the right hand side as an integral with respect to x, and use Fubini's theorem.)

2. Show that if $1 \leq p_0 \leq p_1 \leq \infty$ and $\theta \in [0,1]$, then

$$\|f\|_{L^{p_\theta}} \leq \|f\|_{L^{p_0}}^{1-\theta} \|f\|_{L^{p_1}}^{\theta}$$

whenever f is measurable and

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

This shows $L^{p_0} \cap L^{p_1} \subset L^p$ whenever $1 \leq p_0 \leq p \leq p_1 \leq \infty$. Show also that

$$L^p \subset L^{p_0} + L^{p_1},$$

i.e. every $f \in L^p$ can be decomposed as $f_0 + f_1$, where $f_0 \in L^{p_0}$, $f_1 \in L^{p_1}$.

3. (a) Let $f \in L^p(\mathbb{R}^n)$ for some $p \in [1, \infty)$ and $u(x,y) = f * P_y(x)$ be its Poisson integral. Show that

$$\lim_{y \to 0^+} \|u(x,y) - f(x)\|_{L^p(\mathbb{R}^n)} = 0.$$

(b) More generally, suppose $k(x) \in L^1(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} k(x)dx = 1$, and $k_y(x) = y^{-n} k(y^{-1}x)$ for $y > 0$. Show that we still have

$$\lim_{y \to 0^+} \|f * k_y(x) - f(x)\|_{L^p(\mathbb{R}^n)} = 0.$$

(c) What happens when $p = \infty$?

(Hint: For parts (a) and (b), use the continuity of translations on $L^p(\mathbb{R}^n)$.)

4. For each $q > 0$, let

$$M_q f(x) = \sup_{r>0} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|^q dy \right)^{1/q}$$

where $B(x,r)$ is the ball centered at x and of radius r. Show that M_q is of strong-type (p,p) whenever $p \in (q, \infty]$, and that M_q is of weak-type (q,q).

5. Define the uncentered maximal function

$$M_{\text{uncentered}} f(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f|,$$

where the supremum is over all balls B that contains x. Compare $M_{\text{uncentered}}$ pointwisely to the Hardy-Littlewood maximal function, and discuss the boundedness of this operator on L^p, $1 \leq p \leq \infty$.

6. What if we replace the balls in the definition of the Hardy-Littlewood maximal function (or the uncentered maximal function in the previous question) by cubes of varying side lengths? What if we replace balls by ellipsoids of a fixed eccentricity?

7. A dyadic interval is an interval of the form $2^k m + 2^k[0,1)$ where $k, m \in \mathbb{Z}$. A dyadic cube in $\mathbb{R}^n$ is a cube of the form $2^k m + 2^k[0,1)^n$ where $m \in \mathbb{Z}^n$ and $k \in \mathbb{Z}$ (in other words, products of dyadic intervals of the same lengths). Show that if two dyadic intervals intersect, then one is contained in the other. Similarly, show that if two dyadic cubes in $\mathbb{R}^n$ intersect, then one is contained in another.

8. Define the dyadic maximal function on $\mathbb{R}^n$ by
$$M_{\text{dyadic}} f(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f|$$
where the supremum is over all dyadic cubes in $\mathbb{R}^n$ that contain x. Show that
$$|\{x \in \mathbb{R}^n \colon M_{\text{dyadic}} f(x) > \alpha\}| \leq \frac{1}{\alpha} \|f\|_{L^1(\mathbb{R}^n)},$$
and
$$\|M_{\text{dyadic}} f\|_{L^p(\mathbb{R}^n)} \leq 2(p')^{1/p} \|f\|_{L^p(\mathbb{R}^n)}$$
for all $1 < p \leq \infty$. Note that the constants are independent of the dimension n.

9. A measurable function f on $\mathbb{R}^n$ is said to be in $L \log L$ if
$$\|f\|_{L \log L} := \int_{\mathbb{R}^n} |f(x)| \log(2 + |f(x)|) dx < \infty.$$
It is easy to see that $L \log L \subset L^1$. Let M be the Hardy-Littlewood maximal function on $\mathbb{R}^n$. Show that there exists a constant C_n, such that if f is in $L \log L$ on $\mathbb{R}^n$, then for any set B of finite measure in $\mathbb{R}^n$, we have
$$\|Mf\|_{L^1(B)} \leq |B| + C_n \|f\|_{L \log L}.$$
(Hint: Interpolate between L^1 and L^∞. More precisely,
$$\|Mf\|_{L^1(B)} \leq |B| + \int_B Mf(x) \chi_{Mf>1}(x) dx$$
and
$$\int_B Mf(x) \chi_{Mf \geq 1}(x) dx = \int_0^\infty |\{x \in B \colon Mf(x) \chi_{Mf>1}(x) > \alpha\}| d\alpha.$$
Now just observe that
$$|\{x \in B \colon Mf(x) \chi_{Mf>1}(x) > \alpha\}| = \begin{cases} |\{x \in B \colon Mf(x) > 1\}| & \text{if } \alpha \in (0,1] \\ |\{x \in B \colon Mf(x) > \alpha\}| & \text{if } \alpha \in (1,\infty) \end{cases}$$
But recall that M is weak-type (1,1). In addition (as in the proof of the boundedness of M on L^p for $p \in (1,\infty)$), we have
$$|\{x \in \mathbb{R}^n \colon Mf(x) > \alpha\}| \leq \frac{2C_n}{\alpha} \int_{\mathbb{R}^n} |f| \chi_{|f| > \alpha/2}.$$
Now one just needs to put the estimates together and evaluate an integral in α.)

10. For each measurable subset E of $[-1,1]$ and each $r>0$, let

$$\chi_{E,r}(x) = \frac{1}{2r}\chi_E\left(\frac{x}{r}\right)$$

where χ_E is the characteristic function of the set E. For each $\varepsilon \in (0,1]$, define the small set maximal function $M^{(\varepsilon)}$ by

$$M^{(\varepsilon)}f(x) = \sup_{r>0} \sup_{\substack{E\subset[-1,1] \\ E \text{ measurable} \\ |E|<\varepsilon}} |f| * \chi_{E,r}(x), \quad x \in \mathbb{R}.$$

Show that for every $p \in (1,\infty]$, there exists a constant A_p such that

$$\|M^{(\varepsilon)}f\|_{L^p(\mathbb{R})} \leq A_p \varepsilon^{1-\frac{1}{p}} \|f\|_{L^p(\mathbb{R})} \quad \text{for all } \varepsilon \in (0,1].$$

Extend this result to $\mathbb{R}^n$, $n \geq 1$ as well. (Hint: Interpolate between L^1 and L^∞.)

11. Define, for $f\colon \mathbb{Z} \to \mathbb{C}$, the maximal function

$$M_{\text{discrete}}f(n) = \sup_{N\in\mathbb{N}} \frac{1}{2N+1} \sum_{|m|\leq N} |f(n+m)|.$$

Show that M_{discrete} is of weak-type (1,1), and strong type (p,p) for all $1<p\leq\infty$. (Hint: Compare it to the Hardy-Littlewood maximal function on $\mathbb{R}$.)

12. On $\mathbb{R}^2$, we define the maximal function along rectangles whose sides are parallel to the coordinate axes by

$$M_{\text{parallel}}f(x) = \sup_{\substack{R \text{ rectangle} \\ R \text{ parallel to the axes} \\ x\in R}} \frac{1}{|R|}\int_R |f(y)|dy.$$

Show that M_{parallel} is of strong type (p,p) for all $1<p\leq\infty$. Also show that M_{parallel} is not of weak-type $(1,1)$. (Hint: For the first part, note that M_{parallel} is pointwisely dominated by the composition of the maximal function in the first variable with the second variable. For the second part, we set $f(x)$ to be (a smoothed out version of) the δ function at the origin: then $M_{\text{parallel}}f(x) \geq \frac{c}{|x_1||x_2|}$, which is not in $L^{1,\infty}(\mathbb{R}^2)$.)

13. (a) Suppose K is a compact set, and for every $x \in K$, we are given an open ball $B(x,r_x)$ that is centered at x and of radius r_x. Assume that

$$R := \sup_{x\in K} r_x < \infty.$$

Let $\mathcal{B}$ be this collection of balls, i.e.

$$\mathcal{B} = \{B(x,r_x)\colon x \in K\}.$$

Show that given any $\varepsilon>0$, there exists a finite subcollection $\mathcal{C}$ of balls from $\mathcal{B}$, so that the balls in $\mathcal{C}$ are pairwise disjoint, and so that the (concentric) dilates of balls in $\mathcal{C}$ by $(2+\varepsilon)$ times would cover K.

(b) Let M be the Hardy-Littlewood maximal function on $\mathbb{R}^n$. Using part (a), show that it is weak-type (1,1) with a constant 2^n, i.e.

$$|\{x \in \mathbb{R}^n \colon |Mf(x)| > \alpha\}| \leq \frac{2^n}{\alpha}\|f\|_{L^1(\mathbb{R}^n)}$$

for all $f \in L^1(\mathbb{R}^n)$ and $\alpha > 0$.

14. Let f be a measurable function on $\mathbb{R}^n$. Show that its distribution function $\lambda_f(\alpha) := |\{x \in \mathbb{R}^n \colon |f(x)| > \alpha\}|$, defined for $\alpha \geq 0$, is right-continuous.

15. Let $g \colon (0,\infty) \to [0,\infty)$ be a non-negative decreasing right-continuous function, and $\lambda_g(\alpha) := |\{t \in [0,\infty) \colon g(t) > \alpha\}|$ be its distribution function, defined for $\alpha \in (0,\infty)$.

    (a) Show that g can be reconstructed from λ_g. (Hint: If g were actually strictly decreasing and continuous, then λ_g is essentially the inverse function of g, so g is essentially the inverse function of λ_g. In general, one can verify that g is the distribution function of λ_g.)

    (b) Show that for any $p \in [1,\infty)$, we have

    $$\sup_{\alpha>0} \alpha\lambda_g(\alpha)^{1/p} = \sup_{t>0} t^{1/p} g(t).$$

    (Hint: Suppose $A = \sup_{t>0} t^{1/p} g(t) < \infty$. If $\alpha > 0$ and $\lambda_g(\alpha) > 0$, then let $t = \lambda_g(\alpha) \in (0,\infty)$. We have $g(t-\varepsilon) > \alpha$ for each $\varepsilon > 0$. This shows

    $$\alpha\lambda_g(\alpha)^{1/p} < g(t-\varepsilon)t^{1/p} \leq A\left(1 - \frac{\varepsilon}{t}\right)^{-1/p}$$

    for all $\varepsilon > 0$. Let $\varepsilon \to 0^+$, we see that $\alpha\lambda_g(\alpha)^{1/p} \leq A$, and this shows the left hand side of the desired inequality is bounded by the right hand side there. The reverse inequality is similar since g is the distribution function of λ_g.)

16. Suppose f is a measurable function on $\mathbb{R}^n$, and $1 \leq p < \infty$.

    (a) Show that there exists a unique decreasing non-negative right-continuous function f^* defined on $[0,\infty)$, so that

    $$|\{x \in \mathbb{R}^n \colon |f(x)| > \alpha\}| = |\{t \in [0,\infty) \colon f^*(t) > \alpha\}|$$

    for all $\alpha > 0$. f^* is sometimes called the decreasing rearrangement of f. (Hint: Use part (a) of the previous question.)

    (b) Show that

    $$\sup_{\alpha>0}\left[\alpha|\{x \in \mathbb{R}^n \colon |f(x)| > \alpha\}|^{1/p}\right] = \sup_{t>0} t^{1/p} f^*(t)$$

    and

    $$\sup_{\substack{E \text{ measurable} \\ 0<|E|<\infty}} \frac{1}{|E|^{1/p'}} \int_{\mathbb{R}^n} |f|\chi_E dx = \sup_{0<t<\infty} \frac{1}{t^{1/p'}} \int_0^t f^*(s) ds.$$

    (Hint: Use part (b) of the previous question for the former, and approximate by simple functions for the latter.)

(c) Conclude that if in addition $p \in (1, \infty)$, then

$$\sup_{\alpha>0} \left[\alpha |\{x \in \mathbb{R}^n \colon |f(x)| > \alpha\}|^{1/p}\right] \simeq_p \sup_{\substack{E \text{ measurable} \\ 0<|E|<\infty}} \frac{1}{|E|^{1/p'}} \int_{\mathbb{R}^n} |f| \chi_E dx.$$

(Hint: Use part (b). Note that

$$\frac{1}{t^{1/p'}} \int_0^t f^*(s) ds \geq t^{1/p} f^*(t).$$

Also, if $B = \sup_{t>0} t^{1/p} f^*(t)$, then

$$\frac{1}{t^{1/p'}} \int_0^t f^*(s) ds \leq \frac{1}{t^{1/p'}} \int_0^t B s^{-1/p} ds = Bp'.$$

Hence the desired conclusion.)

(d) Let $1 < p < \infty$. Show that the right hand side of part (c) defines a norm on $L^{p,\infty}(\mathbb{R}^n)$ (in particular, it satisfies a Minkowski inequality).

17. Let $0 < \alpha < n$ and $1 < p < n/\alpha$. From part (d) of the previous question, we saw that $L^{\frac{n}{n-\alpha},\infty}(\mathbb{R}^n)$ admits a norm. Give a direct proof that the Riesz potentials $\mathcal{I}_\alpha$ is of weak-type $(1, \frac{n}{n-\alpha})$ on $\mathbb{R}^n$ using this fact. (Hint: Use Minkowski's inequality for the norm.)

18. Let F, G be two non-negative measurable functions, and $p \in (0, \infty)$.

(a) Suppose we have a pointwise inequality $F(x) \leq c_1 G(x) + c_2 F(x)$ for some constants c_1 and c_2 with $c_2 \leq 1$. Show that

$$\int F(x)^p dx \lesssim \int G(x)^p dx$$

provided that the left hand side is finite.

(b) Suppose there exist constants c_1 and c_2 such that

$$|\{x \colon F(x) > \alpha\}| \leq c_1 |\{x \colon G(x) > c_2 \alpha\}|$$

for all $\alpha > 0$. Show that

$$\int F(x)^p dx \lesssim \int G(x)^p dx.$$

(Hint: Use Question 1.)

(c) Suppose there exist constants a, b, c such that

$$|\{x \colon F(x) > \alpha, G(x) \leq c\alpha\}| \leq a |\{x \colon F(x) > b\alpha\}|$$

for all $\alpha > 0$. If $a < b^p$, show that

$$\int F(x)^p dx \lesssim \int G(x)^p dx$$

provided that the left hand side is finite. (Hint: Indeed

$$|\{x \colon F(x) > \alpha\}| \leq a |\{x \colon F(x) > b\alpha\}| + |\{x \colon G(x) > c\alpha\}|.$$

Now use Question 1.)

(d) Suppose in addition to the conditions in part (c), we also have $b \leq 1$. Then instead of assuming that $\int F(x)^p dx < \infty$, we may assume only that $F \in L^{p_0}$ for some $0 < p_0 \leq p$, and the conclusion of part (c) continues to hold. (Hint: Repeat the proof of part (c), except one considers initially only

$$\int_0^R p\alpha^{p-1} |\{x \colon |F(x)| > \alpha\}| d\alpha;$$

using $b \leq 1$, obtain a bound for I_R uniformly in R before one lets $R \to +\infty$.)

Remark: Inequalities of the kind in parts (c) or (d) are usually called *relative distributional inequalities.*