MATH6081A Homework 4

1. Show that for any $t \in \mathbb{R}$, $(-\Delta)^{it}$ is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$.

2. (a) Let $k \in \mathbb{N}$. Show that the multiplier operator $\partial^\alpha (I - \Delta)^{-k/2}$, with multiplier
$$(2\pi i \xi)^\alpha (1 + 4\pi^2 |\xi|^2)^{-k/2},$$
defines a bounded linear map on $L^p(\mathbb{R}^n)$ whenever α is a multiindex with $|\alpha| \leq k$ and $1 < p < \infty$.

   (b) Similarly, show that for $\alpha > 0$, the multiplier operator $(-\Delta)^{\alpha/2}(I - \Delta)^{-\alpha/2}$, with multiplier
$$(2\pi |\xi|)^\alpha (1 + 4\pi^2 |\xi|^2)^{-\alpha/2},$$
defines a bounded linear map on $L^p(\mathbb{R}^n)$ whenever $1 < p < \infty$.

3. Let $\Phi \in \mathcal{S}(\mathbb{R}^n)$. Let $\Phi_j(x) = 2^{jn}\Phi(2^j x)$ for all $j \in \mathbb{Z}$ and all $x \in \mathbb{R}^n$.

   (a) Show that for every multiindex α, we have
$$\left(\sum_{j \in \mathbb{Z}} |\partial_x^\alpha \Phi_j(x)|^2 \right)^{1/2} \lesssim_\alpha \frac{1}{|x|^{n+|\alpha|}} \quad \text{whenever } x \neq 0.$$
Indeed the sum converges uniformly on compact subsets of $\mathbb{R}^n \setminus \{0\}$. (Hint: First consider the case $|\alpha| = 0$.)

   (b) Show that the ℓ^2 norm in j in part (a) can be replaced by an ℓ^1 norm.

4. Suppose m is a C^∞ function on $\mathbb{R}^n \setminus \{0\}$, such that
$$|\partial_\xi^\alpha m(\xi)| \lesssim_\alpha |\xi|^{-|\alpha|}$$
for all multiindices α and all $\xi \neq 0$.

   (a) Let $\psi(\xi)$ be a smooth function with compact support on the unit ball $B(0,2)$, with $\psi(\xi) \equiv 1$ on $B(0,1)$. Let
$$\varphi(\xi) = \psi(\xi) - \psi(2\xi)$$
so that φ is supported on the annulus $\{1/2 \leq |\xi| \leq 2\}$. For $j \in \mathbb{Z}$, let $K^{(j)} := \mathcal{F}^{-1}[\varphi(2^{-j}\xi)m(\xi)] \in \mathcal{S}(\mathbb{R}^n)$. Show that there exists a C^∞ function $K_0(x)$ on $\mathbb{R}^n \setminus \{0\}$, so that
$$\sum_{|j| \leq J} K^{(j)}(x) \text{ converges uniformly to } K_0(x)$$
on any compact subsets of $\mathbb{R}^n \setminus \{0\}$ as $J \to \infty$. Hence $\mathcal{F}^{-1}m$ agrees with the C^∞ function K_0 away from the origin. (Hint: First show that $|\partial_x^\alpha K^{(j)}(x)| \lesssim_{\alpha,N} 2^{j(n+|\alpha|)} \min\{1, 2^{-jN}|x|^{-N}\}$ for any multiindex α, any $x \in \mathbb{R}^n$, and any positive integer N.)

(b) Let T be the multiplier operator with multiplier m, and K_0 be the C^∞ function on $\mathbb{R}^n \backslash \{0\}$ from part (a). For $\varepsilon > 0$, let $K_\varepsilon(x)$ be the locally integrable function $K_\varepsilon(x) := \chi_{|x|>\varepsilon} K_0(x)$, and let $T_\varepsilon \colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be such that $T_\varepsilon f := f * K_\varepsilon$ for all $f \in \mathcal{S}(\mathbb{R}^n)$. Show that if $f \in \mathcal{S}(\mathbb{R}^n)$, then we have

$$\sup_{\varepsilon>0} |T_\varepsilon f(x)| \lesssim M(Tf)(x) + Mf(x)$$

for every $x \in \mathbb{R}^n$, where M is the Hardy-Littlewood maximal operator. (This is sometimes called *Cotlar's inequality.*) Hence conclude that the map

$$f \mapsto \sup_{\varepsilon>0} |T_\varepsilon f|$$

is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$. This in particular shows that the maximally truncated Hilbert transform

$$H_* f(x) := \sup_{\varepsilon>0} \int_{|y|>\varepsilon} f(x-y) \frac{dy}{y}, \quad x \in \mathbb{R}$$

and the maximally truncated Riesz transform

$$R_* f(x) := \sup_{\varepsilon>0} \int_{|y|>\varepsilon} f(x-y) \frac{y}{|y|^{n+1}} dy, \quad x \in \mathbb{R}^n$$

are bounded on L^p if $1 < p < \infty$. (Hint: For the first part, compare $K_\varepsilon(x)$ to $\sum_{j \leq J} K^{(j)}(x) = \mathcal{F}^{-1}[\psi(2^{-J}\xi) m(\xi)]$ where J is the largest integer for which $2^J \leq \varepsilon^{-1}$. The hint in part (a) helps in this regard.)

Remark. One can also prove that

$$\sup_{\varepsilon>0} |T_\varepsilon f(x)| \lesssim [M(|Tf|^\nu)(x)]^{1/\nu} + Mf(x)$$

for any $0 < \nu < 1$, $f \in C_c^\infty(\mathbb{R}^n)$; see e.g. Duoandikoetxea's *Fourier Analysis*, Lemma 5.15. This shows that $f \mapsto \sup_{\varepsilon>0} |T_\varepsilon f|$ (and hence H_* and R_*) are of weak-type $(1,1)$.

5. Let $\psi(\xi)$ and $\varphi(\xi)$ be as in the previous question. For $f \in \mathcal{S}'(\mathbb{R}^n)$, let

$$P_0 f = \mathcal{F}^{-1}[\psi(\xi)\widehat{f}(\xi)],$$

and

$$\Delta_j f = \mathcal{F}^{-1}[\varphi(2^{-j}\xi)\widehat{f}(\xi)] \quad \text{for } j \in \mathbb{Z}.$$

(a) Show that if $f \in \mathcal{S}(\mathbb{R}^n)$, then

$$P_0 f + \sum_{j=1}^N \Delta_j f \to f$$

in the topology of $\mathcal{S}(\mathbb{R}^n)$ as $N \to \infty$. Hence if $u \in \mathcal{S}'(\mathbb{R}^n)$, then

$$P_0 u + \sum_{j=1}^N \Delta_j u \to u$$

in the topology of $\mathcal{S}'(\mathbb{R}^n)$ as $N \to \infty$. (Hint: It is easier to work on the Fourier side.)

(b) Show that even if $f \in \mathcal{S}(\mathbb{R}^n)$, the sum $\sum_{|j|\leq N} \Delta_j f$ does NOT necessarily converge to f in the topology of $\mathcal{S}(\mathbb{R}^n)$ as $N \to \infty$. Also show that even if $u \in \mathcal{S}'(\mathbb{R}^n)$, the sum $\sum_{|j|\leq N} \Delta_j u$ does NOT necessarily converge to u in the topology of $\mathcal{S}'(\mathbb{R}^n)$ as $N \to \infty$. (Hint: Consider the origin of the Fourier space.)

(c) Show that if $f \in L^p(\mathbb{R}^n)$ and $1 < p < \infty$, then both

$$P_0 f + \sum_{j=1}^{N} \Delta_j f \quad \text{and} \quad \sum_{|j|\leq N} \Delta_j f$$

converges to f in the norm of $L^p(\mathbb{R}^n)$ as $N \to \infty$. (Hint: Use Question 3 of Homework 3 for the first part. For the second part, note that for a Schwartz function g, we have

$$g(x) - \sum_{j\geq -N} \Delta_j g(x) = \int_{\mathbb{R}^n} \widehat{g}(\xi)\psi(2^N\xi)e^{2\pi i x\cdot\xi} d\xi,$$

which tends to zero pointwisely as $N \to \infty$, and is dominated by a constant times $Mg(x)$ which is in L^p when $1 < p < \infty$. Dominated convergence theorem shows that our desired conclusion holds for g in place of f; it now remains to approximate a general L^p function f by a Schwartz function g in L^p norm, and use Young's inequality to handle the error that arises.)

(d) Show that if $f \in L^1(\mathbb{R}^n)$, then

$$P_0 f + \sum_{j=1}^{N} \Delta_j f \to f$$

in the norm of $L^1(\mathbb{R}^n)$ as $N \to \infty$. Also show that even if $f \in L^1(\mathbb{R}^n)$, the sum $\sum_{|j|\leq N} \Delta_j f$ does NOT necessarily converge to f in the topology of $L^1(\mathbb{R}^n)$ as $N \to \infty$. (Hint: Modify your solution to part (c). In particular, the failure in the last part is due essentially to the same reason why the Hardy-Littlewood maximal function is not bounded on L^1: if say $f(x) = \chi_{B(0,1)}(x) \in L^1(\mathbb{R}^n)$, then $|f(x) - \sum_{j\geq -N} \Delta_j f(x)| \gtrsim |x|^{-n}$ as $|x| \to +\infty$.)

(e) Show that even if $f \in L^\infty(\mathbb{R}^n)$, both

$$P_0 f + \sum_{j=1}^{N} \Delta_j f \quad \text{and} \quad \sum_{|j|\leq N} \Delta_j f$$

does NOT necessarily converge to f in the topology of $L^\infty(\mathbb{R}^n)$ as $N \to \infty$. (Hint: For the former, not every bounded function is continuous. For the latter, just take $f = 1$.)

6. (Bernstein's inequality) Let Δ_j be the Littlewood-Paley projection as defined in the previous question. Show that if $f \in \mathcal{S}'(\mathbb{R}^n)$, then

$$\|\Delta_j f\|_{L^q(\mathbb{R}^n)} \lesssim 2^{jn\left(\frac{1}{p}-\frac{1}{q}\right)} \|\Delta_j f\|_{L^p(\mathbb{R}^n)}$$

whenever $j \in \mathbb{Z}$ and $1 \leq p \leq q \leq \infty$. Also, if for $f \in \mathcal{S}'(\mathbb{R}^n)$ and $j \in \mathbb{Z}$, we define $S_j f := \mathcal{F}^{-1}[\psi(2^{-j}\xi)\widehat{f}(\xi)]$ where ψ is as in the last question, then

$$\|S_j f\|_{L^q(\mathbb{R}^n)} \lesssim 2^{jn\left(\frac{1}{p}-\frac{1}{q}\right)}\|S_j f\|_{L^p(\mathbb{R}^n)}$$

whenever $j \in \mathbb{Z}$ and $1 \leq p \leq q \leq \infty$. (Hint: $\Delta_j f = \tilde{S}_j \Delta_j f$ for an appropriate multiplier $\tilde{S}_j$; similarly for $S_j f$.)

7. Let $\Phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \Phi = 0$. For $j \in \mathbb{Z}$ and $f \in \mathcal{S}'(\mathbb{R}^n)$, let

$$\Delta_j f(x) = f * \Phi_j(x) \quad \text{where } \Phi_j(x) = 2^{jn}\Phi(2^j x).$$

Suppose $1 < p \leq 2$, and suppose

$$\left\| \left(\sum_{j \in \mathbb{Z}} |\Delta_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|f\|_{L^p(\mathbb{R}^n)} \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^n).$$

Show that the same remains true for all $f \in L^p(\mathbb{R}^n)$. (Hint: Let $f \in L^p(\mathbb{R}^n)$. We need to show that

$$\left\| \left(\sum_{|j| \leq J} |\Delta_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|f\|_{L^p(\mathbb{R}^n)}$$

with a bound that is uniform in J. But let $\{f_n\}$ be a sequence of Schwartz functions such that $f_n \to f$ in $L^p(\mathbb{R}^n)$. The desired bound hold for f_n in place of f. We want to let $n \to \infty$; for that we need

$$\lim_{n\to\infty} \left\| \left(\sum_{|j| \leq J} |\Delta_j f_n|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} = \left\| \left(\sum_{|j| \leq J} |\Delta_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)}.$$

This holds since

$$\left(\sum_{|j| \leq J} |\Delta_j (f_n - f)|^2 \right)^{1/2} \lesssim \sum_{j \in |J|} |\Delta_j (f_n - f)| \lesssim |J| M(f_n - f),$$

which for every fixed J converges to 0 in $L^p(\mathbb{R}^n)$ as $n \to \infty$.)

8. (Klintchine's inequality) Let $(\Omega, d\mu)$ be a probability space[1]. Let $N \in \mathbb{N}$, and let $\{r_j \colon \Omega \to \{-1,+1\}\}_{1 \leq j \leq N}$ be a sequence of independent random variables; in other words, for every finite sequence $\sigma \in \{-1,+1\}^N$ of signs of length N, we have

$$d\mu\left(\{t \in \Omega \colon r_j(t) = \sigma_j \text{ for all } 1 \leq j \leq N\}\right) = \prod_{j=1}^{N} d\mu\left(\{t \in \Omega \colon r_j(t) = \sigma_j\}\right).$$

[1] a measure space with a non-negative measure $d\mu$ so that $\int_\Omega d\mu = 1$

Assume further that

$$d\mu\left(\{t\in\Omega\colon r_j(t)=+1\}\right)=d\mu\left(\{t\in\Omega\colon r_j(t)=-1\}\right)=\frac{1}{2}$$

for each $1\le j\le N$. Show that for every $p\in(0,\infty)$, there exist constants A_p and B_p that depend only on p (but not on N), such that for every sequence of complex numbers $\{a_j\}_{1\le j\le N}$, we have

$$A_p\left(\sum_{j=1}^N |a_j|^2\right)^{p/2}\le\int_\Omega\left|\sum_{j=1}^N a_j r_j(t)\right|^p d\mu(t)\le B_p\left(\sum_{j=1}^N |a_j|^2\right)^{p/2}.$$

(Hint: Without loss of generality, let $\sum_{j=1}^N |a_j|^2=1$. Let $\mu\in\mathbb{C}$. First show that

$$\int_\Omega\prod_{j=1}^N\left|\exp\left(\mu a_j r_j(t)\right)\right|d\mu(t)=\prod_{j=1}^N\int_\Omega\left|\exp\left(\mu a_j r_j(t)\right)\right|d\mu(t)$$

by using the independence of $\{|\exp(\mu a_j r_j(\cdot))|\colon\Omega\to\{e^{\mu a_j},e^{-\mu a_j}\}\}_{1\le j\le N}$. In other words, compute the integral on the left hand side by considering, for each $\sigma\in\{-1,+1\}^N$, the quantity

$$d\mu\left(\{t\in\Omega\colon|\exp(\mu a_j r_j(t))|=e^{\mu a_j\sigma_j}\text{ for all }1\le j\le N\}\right).$$

Now the above integral identity shows that

$$\int_\Omega\exp\left(\mathrm{Re}\left(\mu\sum_{j=1}^N a_j r_j(t)\right)\right)d\mu(t)=\prod_{j=1}^N\frac{|e^{\mu a_j}|+|e^{-\mu a_j}|}{2}.$$

A similar inequality holds when the real part is replaced by the imaginary part. Thus

$$\int_\Omega e^{|\mu||F(t)|}d\mu(t)\le 2e^{|\mu|^2\sum_{j=1}^N|a_j|^2}=2e^{|\mu|^2}$$

where $F(t):=\left|\sum_{j=1}^N a_j r_j(t)\right|$.

To proceed further, for each $\alpha>0$, set $\mu=\alpha/2$ and use Chebyshev's inequality to obtain

$$d\mu\left(\{t\in\Omega\colon\ |F(t)|>\alpha\}\right)\le 2e^{-\frac{\alpha^2}{4}}.$$

This gives the desired upper bound for $\int_\Omega|F(t)|^p\,d\mu(t)$ for each $p\in(0,\infty)$. To obtain the lower bound, let $p\in(0,\infty)$. Choose $r=r(p)\in(0,\infty)$ and $\theta=\theta(p)\in(0,1)$ such that

$$\frac{1}{2}=\frac{1-\theta}{p}+\frac{\theta}{r}.$$

Then Hölder's inequality gives

$$1=\int_\Omega|F(t)|^2d\mu(t)\le\left(\int_\Omega|F(t)|^p d\mu(t)\right)^{\frac{2(1-\theta)}{p}}\left(\int_\Omega|F(t)|^r d\mu(t)\right)^{\frac{2\theta}{r}}.$$

But we already knew an upper bound for $\int_\Omega|F(t)|^r d\mu(t)$. Invoking that, we obtain a lower bound for $\int_\Omega|F(t)|^p d\mu(t)$ that depends only on p, and this concludes the proof of the desired estimate.)

9. Let $N \in \mathbb{N}$.

   (a) Show that if $\Omega = [-1/2, 1/2]^N$, $d\mu(t) = dt$ and $r_j(t) = \operatorname{sgn}(t_j)$ for $1 \le j \le N$, then the hypothesis in the previous question is satisfied.

   (b) Show that if $\Omega = [0,1)$, $d\mu(t) = dt$ and

$$r_j(t) = \begin{cases} +1 & \text{if } t \in [k2^{-j}, (k+1)2^{-j}) \text{ for some odd integer } k \\ -1 & \text{if } t \in [k2^{-j}, (k+1)2^{-j}) \text{ for some even integer } k \end{cases}$$

   is the j-th Rademacher function on $[0,1]$ for $1 \le j \le N$, then the hypothesis in the previous question is satisfied.

10. Let $N \in \mathbb{N}$, $\Omega = \mathbb{S}^{2N-1} = \{t \in \mathbb{C}^N : |t| = 1\}$, $d\mu(t)$ be the surface measure on $\mathbb{S}^{2N-1}$ induced from the Lebesgue measure on $\mathbb{C}^N$, and $r_j(t) = \overline{t_j}$ for each $t \in \Omega$ and each $1 \le j \le N$. Prove the following variant of the main conclusion of Question 1:

$$\int_\Omega \left| \sum_{j=1}^N a_j r_j(t) \right|^p d\mu(t) = 2|\mathbb{S}^{2N-2}| B\left(\frac{p+1}{2}, N - \frac{1}{2}\right) \left(\sum_{j=1}^N |a_j|^2 \right)^{p/2}$$

whenever $\{a_j\}_{j=1}^N$ is a complex sequence and $0 < p < \infty$; here

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

for $a, b > 0$. (Hint: Use homogeneity to reduce to calculating

$$\int_{\mathbb{S}^{2N-1}} |\operatorname{Re} t_1|^p d\mu(t),$$

which can be shown, via spherical coordinates, to be equal to

$$2|\mathbb{S}^{2N-2}| \int_0^{\frac{\pi}{2}} (\cos\theta)^p (\sin\theta)^{2N-2} d\theta.$$

Now just make a further change of variable $x^2 = \cos\theta$, $1 - x^2 = \sin\theta$.)

11. (Marcinkiewicz-Zygmund inequality) Let $T : L^p \to L^p$ be a bounded linear operator for a certain $p \in [1, \infty)$, with operator norm A. Show that for any sequence of measurable functions $\{f_j\}_{j \in \mathbb{N}}$, we have

$$\left\| \left(\sum_{j=1}^\infty |Tf_j|^2 \right)^{1/2} \right\|_{L^p} \le A \left\| \left(\sum_{j=1}^\infty |f_j|^2 \right)^{1/2} \right\|_{L^p}.$$

(Hint: Fix $N \in \mathbb{N}$. Let $(\Omega, d\mu(t))$ be a measure space and $\{r_j\}_{1 \le j \le N}$ be a sequence of random variables as in Question 1. For each $t \in \Omega$, let $f_t(x) = \sum_{j=1}^N r_j(t) f_j(x)$. We have

$$\int |Tf_t(x)|^p dx \le A^p \int |f_t(x)|^p dx \quad \text{for every } t \in \Omega.$$

Average this inequality using Klintchine's inequality on both sides, and then let $N \to \infty$. Alternatively, let $(\Omega, d\mu(t))$ and $\{r_j\}$ be as in the previous question, carry out a similar argument, and use the identity from the previous question instead. Note that in this latter strategy, the same constant pops up in both sides of the equation when we average, so that it can be cancelled from both sides.)

12. Let $m \colon \mathbb{R}^n \to \mathbb{C}$ be a measurable function that grows at most polynomially at infinity. Let $\varphi(\xi) = \psi(\xi) - \psi(2\xi)$ where $\psi(\xi)$ be a smooth function with compact support on the unit ball $B(0,2)$, with $\psi(\xi) \equiv 1$ on $B(0,1)$. Let $K := \mathcal{F}^{-1}m$, and for $j \in \mathbb{Z}$, let $K^{(j)} := \mathcal{F}^{-1}[\varphi(2^{-j}\xi)m(\xi)]$. Consider the following conditions on m:

    (a) m is C^∞ on $\mathbb{R}^n \setminus \{0\}$, and
    $$|\partial_\xi^\alpha m(\xi)| \lesssim_\alpha |\xi|^{-|\alpha|}$$
    for all $\xi \neq 0$ and all multiindices α.

    (b) m is r-times differentiable $\mathbb{R}^n \setminus \{0\}$, where r is the smallest integer greater than $n/2$, and
    $$R^{-\frac{n}{2}} \left(\int_{R<|\xi|<2R} |\partial_\xi^\alpha m(\xi)|^2 d\xi \right)^{1/2} \lesssim R^{-|\alpha|}$$
    for all multiindices α with $|\alpha| \leq r$ and all $R > 0$.

    (c) There exists $\varepsilon > 0$ and $C > 0$ such that
    $$\int_{|x|\geq\omega} |K^{(j)}(x)| dx \leq C(1 + 2^j\omega)^{-\varepsilon}$$
    for all $\omega \geq 0$ and $j \in \mathbb{Z}$.

    (d) $m(\xi) \in L^\infty(\mathbb{R}^n)$, K agrees with a locally integrable function K_0 on $\mathbb{R}^n \setminus \{0\}$, and there exists $C > 0$ such that
    $$\int_{|x|\geq 2|y|} |K_0(x-y) - K_0(x)| dx \leq C$$
    for all $y \in \mathbb{R}^n$.

    Show that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d). In particular, if any of these conditions are satisfied, then m defines a multiplier that is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$, and is of weak-type (1,1).
    (Hint: For (b) $\Rightarrow$ (c), write $|K^{(j)}(x)|$ as $(1+|x|)^{-r}(1+|x|)^r|K^{(j)}(x)|$ and apply Cauchy-Schwarz. For (c) $\Rightarrow$ (d), use $K(x) = \sum_{j\in\mathbb{Z}} K^{(j)}(x)$ and split the sum into two, depending on whether $2^j \gtrsim |y|^{-1}$ or $2^j \lesssim |y|^{-1}$. In the latter case, one should estimate the derivative of $K^{(j)}$, using the observation that if (c) holds, then the inequality there also holds if we replace φ with a suitable multiple of φ in the definition of the $K^{(j)}$'s.)

13. Establish the following variant of the vector-valued singular integral theorem, which makes more explicit the dependence of the operator norms on various parameters: Let B_1, B_2 be Banach spaces. Let $\mathrm{End}(B_1, B_2)$ be the space of continuous endomorphisms from B_1 to B_2. Let $L^p(\mathbb{R}^n, B_j)$ be the space of L^p mappings from $\mathbb{R}^n$ into

B_j. Write $(\mathbb{R}^n \times \mathbb{R}^n)^*$ for the set $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\}$. Let T be a bounded linear operator from $L^q(\mathbb{R}^n, B_1)$ to $L^q(\mathbb{R}^n, B_2)$ for some $q \in (1,\infty]$, with

$$\|T\|_{L^q(\mathbb{R}^n,B_1)\to L^q(\mathbb{R}^n,B_2)} \leq 1.$$

Suppose there exists a function

$$K_0(x,y) \in L^\infty_{\text{loc}}((\mathbb{R}^n \times \mathbb{R}^n)^*, \text{End}(B_1, B_2)),$$

such that

$$Tf(x) = \int_{\mathbb{R}^n} K_0(x,y) f(y) dy$$

for every $f \in L^1(\mathbb{R}^n, B_1)$ with compact support, and for a.e. $x \notin \text{supp}(f)$. Suppose in addition that

$$\sup_{(y,y_0)\in\mathbb{R}^n\times\mathbb{R}^n} \int_{|x-y_0|\geq 2|y-y_0|} \|K_0(x,y) - K_0(x,y_0)\|_{\text{End}(B_1,B_2)} dx \leq A$$

for some $A \geq 1$. Then T extends as a continuous linear operator from $L^1(\mathbb{R}^n, B_1)$ to $L^{1,\infty}(\mathbb{R}^n, B_2)$ with norm $\lesssim A$, and a continuous linear operator from $L^p(\mathbb{R}^n, B_1)$ to $L^p(\mathbb{R}^n, B_2)$ for all $1 < p < q$ with norm $\lesssim A^{q'(\frac{1}{p}-\frac{1}{q})}$.

14. Suppose $\{k_j\}_{j\in\mathbb{Z}}$ is a sequence of non-negative integrable functions on $\mathbb{R}^n$, with

$$\sup_{j\in\mathbb{Z}} \|k_j\|_{L^1(\mathbb{R}^n)} \leq 1$$

and

$$\sup_{x\in\mathbb{R}^n} \int_{|y|\geq 2|x|} \sup_{j\in\mathbb{Z}} |k_j(y-x) - k_j(y)| dy \leq A$$

for some constant $A \geq 1$.

(a) Suppose $1 < p < \infty$, and that $\{f_j\}_{j\in\mathbb{Z}}$ is a sequence of measurable functions on $\mathbb{R}^n$ with $\|\|f_j\|_{\ell^2(\mathbb{Z})}\|_{L^p(\mathbb{R}^n)} < \infty$. Show that $\|f_j * k_j\|_{\ell^2(\mathbb{Z})} \in L^p(\mathbb{R}^n)$, and that

$$\|\|f_j * k_j\|_{\ell^2(\mathbb{Z})}\|_{L^p(\mathbb{R}^n)} \lesssim A^{2\left|\frac{1}{2}-\frac{1}{p}\right|} \|\|f_j\|_{\ell^2(\mathbb{Z})}\|_{L^p(\mathbb{R}^n)}.$$

(Hint: By monotone convergence, we may assume that only finitely many f_j's are non-zero, say $f_j \equiv 0$ whenever $|j| \geq N$. For $k \in \mathbb{N}$ and $|j| < N$, choose $f_j^{(k)} \in \mathcal{S}(\mathbb{R}^n)$ such that $\|\|f_j^{(k)} - f_j\|_{\ell^2_j}\|_{L^p_x} \to 0$ as $k \to +\infty$. Then $f_j^{(k)} * k_j \to f_j * k_j$ in $L^p(\mathbb{R}^n)$ for all $|j| < N$ as $k \to \infty$, so by passing to a subsequence, we have pointwise a.e. convergence of $\|f_j^{(k)} * k_j\|_{\ell^2_j}$ to $\|f_j * k_j\|_{\ell^2_j}$ on $\mathbb{R}^n$. Now by Fatou's lemma, we only need to bound

$$\|\|f_j^{(k)} * k_j\|_{\ell^2_j}\|_{L^p_x}$$

uniformly in k. But this follows from Fubini if $p = 2$, and follows from the vector-valued singular integral theorem and duality for other $1 < p < \infty$.)

(b) (Zó's lemma) Show that the associated maximal function

$$\mathfrak{M}f(x) := \sup_{j\in\mathbb{Z}} |f| * k_j(x)$$

is of weak-type $(1,1)$, and is bounded on L^p for all $1 < p \leq \infty$; more precisely,

$$\|\mathfrak{M}f\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim A\|f\|_{L^1(\mathbb{R}^n)},$$

and

$$\|\mathfrak{M}f\|_{L^p(\mathbb{R}^n)} \lesssim A^{1/p}\|f\|_{L^p(\mathbb{R}^n)}$$

for all $1 < p \leq \infty$. (Hint: First establish the case $p = \infty$. Then use the theorem about vector-valued singular integrals to establish the remaining cases.)

15. Suppose $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ is a non-negative integrable function on $\mathbb{R}^n$ satisfying

$$\int_{\mathbb{R}^n} \varphi(y)dy \lesssim 1,$$

$$\int_{|y|\geq R} \varphi(y)dy \lesssim R^{-1} \quad \text{for all } R \geq 1,$$

and

$$\int_{\mathbb{R}^n} |\varphi(y-x) - \varphi(y)|dy \lesssim |x| \quad \text{for all } x \in \mathbb{R}^n \text{ with } |x| \geq 1.$$

(a) Define, for all $j \in \mathbb{Z}$ and all $x \in \mathbb{R}^n$,

$$\varphi_j(x) = 2^{jn}\varphi(2^j x).$$

Show that

$$\sup_{x\in\mathbb{R}^n} \int_{|y|\geq 2|x|} \sup_{j\in\mathbb{Z}} |\varphi_j(y-x) - \varphi_j(y)|dy \lesssim 1.$$

It follows, from the previous question, that the maximal function

$$f \mapsto \sup_{j\in\mathbb{Z}} |f| * \varphi_j$$

is of weak-type $(1,1)$ and of strong-type (p,p) for all $1 < p \leq \infty$, and that

$$\|\|f_j * \varphi_j\|_{\ell^2(\mathbb{Z})}\|_{L^p(\mathbb{R}^n)} \lesssim \|\|f_j\|_{\ell^2(\mathbb{Z})}\|_{L^p(\mathbb{R}^n)}$$

for all $1 < p < \infty$ whenever $\{f_j\}_{j\in\mathbb{Z}}$ is a sequence of measurable functions on $\mathbb{R}^n$ for which the right hand side is finite.

(b) More generally, fix $r \in \mathbb{R}^n$ with $|r| \geq 2$. Define, for all $j \in \mathbb{Z}$ and all $x \in \mathbb{R}^n$,

$$k_j(x) = 2^{jn}\varphi(2^j x + r).$$

Show that

$$\sup_{x\in\mathbb{R}^n} \int_{|y|\geq 2|x|} \sup_{j\in\mathbb{Z}} |k_j(y-x) - k_j(y)|dy \lesssim \log|r|.$$

What does this say about the maximal function $f \mapsto \sup_{j\in\mathbb{Z}} |f| * k_j$, and the norm $\|\|f_j * k_j\|_{\ell^2(\mathbb{Z})}\|_{L^p(\mathbb{R}^n)}$ for $1 < p < \infty$?

16. Show that if $\varphi(x) = \chi_{B(0,1)}(x)$ is the characteristic function of the unit ball on $\mathbb{R}^n$ centered at the origin, then φ satisfies the hypothesis of the previous question. The same is true if $\varphi(x)$ is the absolute value of any Schwartz function on $\mathbb{R}^n$. In particular, if the Δ_j's are the Littlewood-Paley projections defined in Question 7, then
$$\| \|\Delta_j f_j\|_{\ell^2(\mathbb{Z})} \|_{L^p(\mathbb{R}^n)} \lesssim \| \|f_j\|_{\ell^2(\mathbb{Z})} \|_{L^p(\mathbb{R}^n)}$$
for $1 < p < \infty$. Can you give a more direct proof of this last inequality, that uses Question 3 instead of the estimate on $\sup_{x \in \mathbb{R}^n} \int_{|y| \geq 2|x|} \sup_{j \in \mathbb{Z}} |\Phi_j(y - x) - \Phi_j(y)| dy$?