MATH6081A Homework 5

1. Suppose $\gamma > 0$ and $f \in \mathcal{S}'(\mathbb{R}^n)$. Show that $f \in \Lambda^\gamma$, if and only if there exists a sequence of C^∞ functions $\{f_j\}_{j \geq 0}$ on $\mathbb{R}^n$, such that

$$\|\partial^\beta f_j\|_{L^\infty(\mathbb{R}^n)} \lesssim_\beta 2^{-j\gamma} 2^{j|\beta|} \quad \text{for all } j \geq 0 \text{ and all multiindices } \beta,$$

and

$$f = \sum_{j \geq 0} f_j$$

with convergence in the topology of $\mathcal{S}'(\mathbb{R}^n)$. (Hint: Adapt an argument from the lecture notes. You will also need to use the fact that the Littlewood-Paley projections P_k $(k \geq 1)$ can be chosen to be convolutions against dilations of a Schwartz function Φ with zero moments, i.e. one that satisfies $\int_{\mathbb{R}^n} y^\beta \Phi(y) dy = 0$ for all monomials y^β. Then in estimating

$$P_k f_j(x) = \int_{\mathbb{R}^n} f_j(x-y) 2^{kn} \Phi(2^k y) dy,$$

you may Taylor expand $f_j(x-y)$ in y up to order $\lfloor \gamma \rfloor$, and proceed as in the lecture.)

2. (a) Show that $\Lambda^\gamma(\mathbb{R}^n)$ embeds continuously into $\Lambda^\beta(\mathbb{R}^n)$ if $0 < \beta < \gamma$.

   (b) Show that $W^{\alpha,p}(\mathbb{R}^n)$ embeds continuously into $W^{\beta,q}(\mathbb{R}^n)$ if

$$0 \leq \beta < \alpha, \quad 1 < p \leq q < \infty, \quad \text{and} \quad \frac{1}{q} \leq \frac{1}{p} - \frac{\alpha - \beta}{n}.$$

   (c) Show that $W^{\alpha,\frac{n}{\alpha}}(\mathbb{R}^n)$ does not embed into $L^\infty(\mathbb{R}^n)$ if $0 < \alpha < n$. (Hint: An illustrative way of constructing such a function is to take

$$f(x) = \sum_{k=1}^\infty \frac{1}{k} g(2^{k+1} x),$$

   where $g \in \mathcal{S}(\mathbb{R}^n)$ is a function such that $\widehat{g}$ is supported on an annulus $|\xi| \simeq 1$, and $g(0) = 1$. Then one easily checks that $f \notin L^\infty(\mathbb{R}^n)$ because it is unbounded near the origin. But if $I = \sum_{j=0}^\infty P_j$ is the standard Littlewood-Paley decomposition, then we have

$$\sum_{j=0} \left(2^{j\alpha} |P_j f(x)|\right)^2 \lesssim \begin{cases} |x|^{-2\alpha} \log(\frac{1}{|x|})^{-2} & \text{for } |x| \leq 1/2, \\ |x|^{-N} & \text{for any } N \text{ for } |x| \geq 1/2. \end{cases}$$

   Since $0 < \alpha < n$, one can raise both sides to power $\frac{n}{2\alpha}$, and integrate; this shows $f \in W^{\alpha,\frac{n}{\alpha}}(\mathbb{R}^n)$. Alternatively, take $f(x) = \log\log\frac{1}{|x|}$ if $|x| < e^{-1}$, and zero otherwise, and mimic the above strategy. One can also avoid the use of Littlewood-Paley decompositions if α were a positive integer.)

3. Suppose $k \in \mathbb{N}$ and $1 \leq p < \infty$. Show that $C_c^\infty(\mathbb{R}^n)$ is dense in the Sobolev space $W^{k,p}(\mathbb{R}^n)$. (Hint: For $f \in W^{k,p}(\mathbb{R}^n)$, consider spatial cut-offs of $f * \Psi_\varepsilon(x)$ where $\Psi \in \mathcal{S}(\mathbb{R}^n)$ with $\int \Psi = 1$, and $\Psi_\varepsilon(x) = \varepsilon^{-n} \Psi(\varepsilon^{-1} x)$.)

4. (a) Suppose $f \in C_c^\infty(\mathbb{R}^n)$, $n \geq 2$. Show that

$$|f(x)| \lesssim \mathcal{I}_1(|\nabla f|)(x)$$

for any $x \in \mathbb{R}^n$. Hence if $k \in \mathbb{N}$, $1 \leq k < n$, then $|f(x)| \lesssim \mathcal{I}_k(|\nabla^k f|)(x)$ for any $x \in \mathbb{R}^n$. (Hint: $f(x) = \int_0^\infty \omega \cdot \nabla f(x - t\omega) dt$ for every $x \in \mathbb{R}^n$ and $\omega \in \mathcal{S}^{n-1}$. Average over $\omega \in \mathcal{S}^{n-1}$ and change coordinates $y = t\omega$.)

(b) Use this and the mapping properties of $\mathcal{I}_k$ to give an alternative proof of the Sobolev embedding theorem for $W^{k,p}(\mathbb{R}^n)$ if $1 < p < n/k$.

(c) The above proof can be modified to give an alternative proof of the Sobolev embedding theorem in the case $p = 1$. The key is to show that $W^{1,1}(\mathbb{R}^n)$ embeds into $L^{n/(n-1)}$. To do so, let $f \in C_c^\infty(\mathbb{R}^n)$. For $j \in \mathbb{Z}$, let g_j be the continuous function given by

$$g_j(x) = \max\{\min\{|f(x)|, 2^{j+1}\}, 2^{j-1}\} - 2^{j-1}.$$

Then the distributional derivative of g_j is given by

$$\nabla g_j(x) = \begin{cases} \nabla|f(x)| & \text{if } 2^{j-1} < |f(x)| < 2^{j+1} \\ 0 & \text{otherwise} \end{cases}.$$

Show first

$$2^{\frac{j(n-1)}{n}} |\{x \in \mathbb{R}^n : g_j(x) > 2^{j-1}\}| \lesssim \|\nabla g_j\|_{L^1} \lesssim \int_{2^{j-1}<|f(x)|<2^{j+1}} |\nabla f(x)| dx$$

using that $\mathcal{I}_1$ is of weak-type $(1, \frac{n}{n-1})$. Then sum over $j \in \mathbb{Z}$, noting that $\{x \in \mathbb{R}^n : 2^j < |f(x)| \leq 2^{j+1}\} \subset \{x \in \mathbb{R}^n : g_j(x) > 2^{j-1}\}$.

(d) Show that if C_n is the best constant of the Gagliardo-Nirenberg inequality in $\mathbb{R}^n$, $n \geq 2$, i.e. if C_n is the smallest constant such that

$$\|f\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq C_n \|\nabla f\|_{L^1(\mathbb{R}^n)}$$

for all $f \in C_c^\infty(\mathbb{R}^n)$, then for any bounded smooth domain $\Omega \subset \mathbb{R}^n$, we have

$$|\Omega|^{\frac{n-1}{n}} \leq C_n |\partial\Omega|.$$

(Hint: Approximate the characteristic function of Ω by $C_c^\infty(\mathbb{R}^n)$.)

5. (a) Suppose $f \in C^\infty$ on some open ball $B \subset \mathbb{R}^n$, $n \geq 2$. Show that

$$|f(x) - f_B| \lesssim \mathcal{I}_1(|\nabla f|\chi_B)(x)$$

for any $x \in B$. Here $f_B := ⨍_B f$ is the average of f on B, and χ_B is the characteristic function of B. (Hint: Compute $f(z) - f(x)$ for $z \in B$ by writing $z = x + t\omega$ with $t > 0$, $\omega \in \mathcal{S}^{n-1}$ and applying the fundamental theorem of calculus in the t variable. Then average over $z \in B$.)

(b) Hence give an alternative proof of Morrey's embedding theorem for $W^{k,p}(\mathbb{R}^n)$ when $p \in (n/k, \infty)$. (Hint: Without loss of generality, assume $k = 1$ and $p \in (n, \infty)$. Then

$$|f(x) - f(y)| \leq |f(x) - f_B| + |f(y) - f_B|$$

where B a ball of radius $\simeq |x - y|$ containing both x and y. Then apply the above estimate, and estimate the right-hand side using Hölder's inequality and the assumption that $\|\nabla f\|_{L^p} < \infty$ where $p > n$.)

(c) Prove also Poincare's inequality: if $f \in C^\infty$ on some open ball $B \subset \mathbb{R}^n$, $n \geq 1$, then

$$\|f(x) - f_B\|_{L^p(B)} \lesssim r\|\nabla f\|_{L^p(B)}, \quad 1 \leq p \leq \infty,$$

where r is the radius of the ball B. (Hint: The case $n \geq 2$ follows from part (a) and Young's convolution inequality. Find a substitute of the proof in the case $n = 1$.)

(d) Using the Poincare inequality in (c), give an alternative proof that $W^{1,n}(\mathbb{R}^n)$ embeds into BMO. This also implies that $W^{\alpha,\frac{n}{\alpha}}(\mathbb{R}^n)$ embeds into BMO for all $\alpha \in (1, n)$ by Question 2(b). (Hint: For any ball $B \subset \mathbb{R}^n$, we have

$$\fint_B |f(x) - f_B| dx \leq \left(\fint_B |f(x) - f_B|^n dx\right)^{1/n} \lesssim r \left(\fint_B |\nabla f(x)|^n dx\right)^{1/n} \leq \|\nabla f\|_{L^n(\mathbb{R}^n)}$$

where we have applied Poincare with $p = n$ in the second inequality.)

6. (a) Show that $\log|x|$ is a BMO function on $\mathbb{R}^n$ for all $n \geq 1$. (Hint: If $|x_0| \leq 2r$, then $B(x_0, r) \subset B(0, 3r)$, and

$$\fint_{B(0,3r)} |\log|x| - \log r| dx$$

is a finite constant independent of r; on the other hand, if $|x_0| \geq 2r$, then $|x_0|/2 \leq |x| \leq 2|x_0|$ for all $x \in B(x_0, r)$, so

$$\fint_{B(x_0,r)} |\log|x| - \log|x_0|| dx$$

is bounded by $\log 2$.)

(b) Show that

$$f(x) = \begin{cases} \log x & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

is not a BMO function on $\mathbb{R}$. (Hint: Consider $\fint_B |f(x) - f_B| dx$ where $B = (-r, r)$ and $r \to +\infty$.)

7. Let H be the Hilbert transform on $\mathbb{R}$.

(a) Show that H is not bounded on L^1. (Hint: Consider a sequence of L^1 functions converging to the δ function in $\mathcal{S}'(\mathbb{R}^n)$.)

(b) Hence deduce that H is not bounded on L^∞. (Hint: Use duality.)

(c) Show that if H is extended as a continuous linear operator from L^∞ to BMO and $f(x) = \text{sgn}(x)$, then $Hf(x) = \frac{2}{\pi} \log |x|$. This provides a direct proof that H is not bounded on L^∞.

8. (a) Show that we could have used cubes instead of balls in the definition of BMO.

(b) Show that if f is a BMO function, then so is $|f|$.

(c) Show that if f, g are both BMO functions, then so is $\max\{f, g\}$ and $\min\{f, g\}$.

(d) Show that if f is a BMO function, then

$$\fint_{B_R} |f(x) - f_{B_1}| dx \lesssim (\log R) \|f\|_{BMO} \quad \text{for every } R \geq 10,$$

and

$$\int_{\mathbb{R}^n} \frac{|f(x) - f_{B_1}|}{(1+|x|)^{n+\varepsilon}} dx \lesssim \|f\|_{BMO} \quad \text{for every } \varepsilon > 0.$$

In particular, if f is a BMO function, then

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|)^{n+\varepsilon}} < \infty \quad \text{for every } \varepsilon > 0,$$

so every BMO function on $\mathbb{R}^n$ is 'almost in L^∞', and defines a tempered distribution on $\mathbb{R}^n$. (Hint: First recall

$$\fint_{B_{2^k}} |f(x) - f_{B_{2^k}}| dx \leq \|f\|_{BMO}$$

for all non-negative integers k; here B_{2^k} is the ball of radius 2^k centered at the origin. Then compare this inequality with the one for $B_{2^{k-1}}$, to see that

$$|f_{B_{2^k}} - f_{B_{2^{k-1}}}| \lesssim \|f\|_{BMO}.$$

Hence

$$|f_{B_{2^k}} - f_{B_1}| \lesssim k \|f\|_{BMO}.$$

This easily gives the first desired inequality when $R = 2^k$ for some positive integer k; the second desired inequality then follows by dividing $\mathbb{R}^n \setminus B_1$ into a union of dyadic annuli.)

9. The goal of this question is to prove the John-Nirenberg inequalities for BMO functions on $\mathbb{R}^n$.

(a) Show that there exists constants C_1, C_2 depending only on n, such that for any BMO function f on $\mathbb{R}^n$, and any cube $Q \subset \mathbb{R}^n$, we have

$$\frac{|\{x \in Q \colon |f(x) - f_Q| > \lambda\}|}{|Q|} \leq C_1 e^{-\frac{C_2 \lambda}{\|f\|_{BMO}}}$$

for all $\lambda > 0$. Here $f_Q := \fint_Q f$ is the average of f on Q. (Hint: Without loss of generality, assume $\|f\|_{BMO} = 1$. Then in particular $\fint_Q |f(x) - f_Q| dx \leq 1$.

Now perform a Calderon-Zygmund decomposition of $f - f_Q$ at height 2, by repeatedly bisecting Q into 2^n congruent sub-cubes, and keeping only those over which $|f - f_Q|$ has average > 2. Then we obtain a family of cubes $\{Q_{j_1}\}$ so that

$$2 < \fint_{Q_{j_1}} |f - f_Q| \leq 2^{n+1} \quad \text{for all } j_1$$

and

$$|f(x) - f_Q| \leq 2 \quad \text{for a.e. } x \in Q \setminus \bigcup_{j_1} Q_{j_1}.$$

Now for each j_1, perform a Calderon-Zygmund decomposition of $f - f_{Q_{j_1}}$ at height 2, by repeatedly bisecting Q_{j_1} into 2^n congruent sub-cubes, and keeping only those over which $|f - f_{Q_{j_1}}|$ has average > 2. Then we obtain a family of cubes $\{Q_{j_1,j_2}\}$ so that

$$2 < \fint_{Q_{j_1,j_2}} |f - f_{Q_{j_1}}| \leq 2^{n+1} \quad \text{for all } j_1, j_2$$

and

$$|f(x) - f_{Q_{j_1}}| \leq 2 \quad \text{for a.e. } x \in Q_{j_1} \setminus \bigcup_{j_2} Q_{j_1,j_2}$$

for every j_1. Repeat this process k times, and we obtain a family of cubes $\{Q_{j_1,j_2,\dots,j_k}\}$ so that

$$2 < \fint_{Q_{j_1,j_2,\dots,j_k}} |f - f_{Q_{j_1,j_2,\dots,j_{k-1}}}| \leq 2^{n+1} \quad \text{for all } j_1, j_2, \dots, j_k \tag{1}$$

and

$$|f(x) - f_{Q_{j_1,j_2,\dots,j_{k-1}}}| \leq 2 \quad \text{for a.e. } x \in Q_{j_1,j_2,\dots,j_{k-1}} \setminus \bigcup_{j_k} Q_{j_1,j_2,\dots,j_k} \tag{2}$$

for every $j_1, j_2, \dots, j_{k-1}$. (1) shows that

$$|f_{Q_{j_1,j_2,\dots,j_k}} - f_{Q_{j_1,j_2,\dots,j_{k-1}}}| \leq 2^{n+1}$$

for any $j_1, \dots, j_k$, so iterating this gives

$$|f_{Q_{j_1,j_2,\dots,j_k}} - f_Q| \leq k2^{n+1}$$

for any $j_1, \dots, j_k$. Together with (2) we get

$$|f(x) - f_Q| \leq k2^{n+1} \quad \text{for a.e. } x \in Q \setminus \bigcup_{j_1,\dots,j_k} Q_{j_1,j_2,\dots,j_k}.$$

But (1) also shows that

$$\sum_{j_k} |Q_{j_1,\dots,j_k}| \leq \frac{1}{2} \sum_{j_k} \int_{Q_{j_1,\dots,j_k}} |f - f_{Q_{j_1,j_2,\dots,j_{k-1}}}| \leq \frac{1}{2} |Q_{j_1,j_2,\dots,j_{k-1}}|,$$

so inductively we have

$$\sum_{j_1,\dots,j_k} |Q_{j_1,\dots,j_k}| \leq \frac{1}{2^k} |Q|.$$

Now it suffices to choose k so that $\lambda \simeq k2^{n+1}$ (which is possible if $\lambda > 2^{n+1}$; if not the estimate is trivial). This finishes the desired estimate.)

(b) Show that for any $p \in (1, \infty)$, there exists a constant $C_{n,p}$, such that

$$\sup_Q \left(\fint_Q |f(y) - f_Q|^p dy \right)^{1/p} \leq C_{n,p} \|f\|_{BMO}$$

for every BMO function f on $\mathbb{R}^n$, where the supremum is over all cubes $Q \subset \mathbb{R}^n$. This is remarkable because the right hand side is certainly bounded by the left hand side, by Hölder's inequality. This inequality in question is thus sometimes called a reversed Hölder inequality. (Hint: Use

$$\fint_Q |f(y) - f_Q|^p dy = \int_0^\infty p\lambda^{p-1} \frac{|\{x \in Q \colon |f(x) - f_Q| > \lambda\}|}{|Q|} d\lambda$$

and use part (a). This shows $C_{n,p}^p \leq C_1 \Gamma(p+1) C_2^{-p}$ where C_1, C_2 are as in part (a).)

(c) Show that there exists constants $c, C > 0$ depending only on n, such that for every BMO function f on $\mathbb{R}^n$ and every cube $Q \subset \mathbb{R}^n$, we have

$$\fint_Q \exp\left(\frac{c|f(y) - f_Q|}{\|f\|_{BMO}} \right) dy \leq C.$$

(Hint: Expand exp as a power series, and apply (b) with $p = m$ for every positive integer m. We also need the explicit bound for $C_{n,p}$ as in the hint of part (b). Indeed, we use that

$$\sum_{m=1}^\infty \frac{C_1 \Gamma(m+1) C_2^{-m} c^m}{m!} = \sum_{m=1}^\infty C_1 C_2^{-m} c^m$$

which is finite if $c < C_2$.)

10. The Moser-Trudinger inequality states that if $n \geq 1$, and $\omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the surface area of $\mathbb{S}^{n-1}$, then there exists a constant C depending only on n, such that for every ball $B \subset \mathbb{R}^n$ and every $f \in C_c^\infty(B)$, we have

$$\fint_B \exp\left(n\omega_{n-1}^{\frac{1}{n-1}} \left(\frac{|f(x)|}{\|\nabla f\|_{L^n(\mathbb{R}^n)}} \right)^{\frac{n}{n-1}} \right) dx \leq C.$$

Moreover the constant $n\omega_{n-1}^{\frac{1}{n-1}}$ is sharp, in the sense that the inequality is false if that were replaced by any larger constant. The goal of this question is to prove a version of this inequality with a non-sharp constant.
Suppose $0 < \alpha < n$. We will prove that there exist constants μ, C depending only on α and n, such that for every ball $B \subset \mathbb{R}^n$ and every $g \in C_c^\infty(B)$, we have

$$\fint_B \exp\left(\mu \left(\frac{|\mathcal{I}_\alpha g(x)|}{\|g\|_{L^{\frac{n}{\alpha}}(B)}} \right)^{\frac{n}{n-\alpha}} \right) dx \leq C. \tag{3}$$

Without loss of generality, assume that B has radius 1/2.

(a) Show that for every $q \in [\frac{n}{\alpha} - 1, \infty)$, we have

$$\int_B |\mathcal{I}_\alpha g(x)|^{\frac{qn}{n-\alpha}} dx \leq \left(\frac{\omega_{n-1}}{n}(q+1)\right)^{q+1} \|g\|_{L^{\frac{n}{\alpha}}(B)}^{\frac{qn}{n-\alpha}}.$$

(Hint: Use Young's convolution inequality.)

(b) Show that (3) holds if μ is chosen sufficiently small. (Hint: Expand exp in power series, use (a) and Stirling's formula.)

11. (a) Let $n \in \mathbb{N}$, $\alpha > 0$, $p \in (\frac{n}{\alpha}, \infty)$. Show that if $\{f_n\}$ is a bounded sequence in $W^{\alpha,p}(\mathbb{R}^n)$, then there exists a subsequence that converges uniformly on compact subsets of $\mathbb{R}^n$. (Hint: Use Morrey's embedding and the Arzela-Ascoli theorem.)

(b) Let $n \in \mathbb{N}$, $1 \leq p < n$ and $1 \leq q < p^*$ where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$. Let Ω be a bounded subset of $\mathbb{R}^n$. Show that if $\{f_n\}$ is a bounded sequence in $W^{1,p}(\mathbb{R}^n)$, then there exists a subsequence that converges in $L^q(\Omega)$. This is called the Rellich compactness theorem. (Hint: By interpolation and Sobolev embedding, it suffices to prove the case $q = 1$. Fix $\varepsilon > 0$. We want to show that there exists $N \in \mathbb{N}$, such that

$$\|f_n - f_m\|_{L^1(\Omega)} < \varepsilon$$

whenever $m, n \geq N$. To do so, fix $\Psi \in C_c^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \Psi = 1$, let $\Psi_\delta(x) = \delta^{-n}\Psi(\delta^{-1}x)$ for $\delta > 0$, and choose δ so small so that

$$\|f_n - f_{n,\delta}\|_{L^1(\Omega)} < \varepsilon/2,$$

where $f_{n,\delta} := f_n * \Psi_\delta$. This is possible because

$$|f_n(x) - f_{n,\delta}(x)| \leq \int_{\mathbb{R}^n} |\Psi(y)||f(x) - f(x-\delta y)| dy \leq \int_{\mathbb{R}^n} \int_0^{\delta|y|} |\Psi(y)||\nabla f|(x - t\frac{y}{|y|}) dt dy$$

which we may integrate over $x \in \Omega$; the smallness comes from the smallness of δ. Now for this fixed δ, $\{f_{n,\delta}\}$ is a sequence of uniformly bounded and equicontinuous functions on Ω; indeed

$$\sup_{n\in\mathbb{N}} \left(\|f_{n,\delta}\|_{L^\infty(\Omega)} + \|\nabla f_{n,\delta}\|_{L^\infty(\Omega)}\right) \leq C_\delta \sup_{n\in\mathbb{N}} \|f_{n,\delta}\|_{L^1(\Omega)} \lesssim C_\delta \sup_{n\in\mathbb{N}} \|f_n\|_{W^{1,p}(\mathbb{R}^n)}.$$

One then concludes the proof using Arzela-Ascoli.)

12. Prove that if $n \in \mathbb{N}$, $1 < p < \infty$ and $\alpha > \frac{n}{p}$, then $W^{\alpha,p}(\mathbb{R}^n)$ is an algebra, and that

$$\|fg\|_{W^{\alpha,p}(\mathbb{R}^n)} \lesssim \|f\|_{W^{\alpha,p}(\mathbb{R}^n)} \|g\|_{W^{\alpha,p}(\mathbb{R}^n)}.$$

This is a simple version of the fractional Leibniz rule. (Hint: Let $I = \sum_{j=0}^\infty P_j$ be a standard Littlewood-Paley decomposition, so that if $f \in W^{\alpha,p}(\mathbb{R}^n)$, then the frequency support of $P_0 f$ is in $|\xi| \leq 2$, and the frequency support of $P_j f$ is in $2^{j-1} \leq |\xi| \leq 2^{j+1}$ for all $j \geq 1$. Then for $j \geq 1$,

$$P_j(fg) = \sum_{k,\ell \geq 0} P_j((P_k f)(P_\ell g)).$$

Many terms in the double sum are zero because of the frequency support considerations. The only ones that survive are the following cases:

(i) $k \simeq j$, $\ell \lesssim j$

(ii) $k \lesssim j$, $\ell \simeq j$

(iii) $k \simeq \ell >> j$

These are known as high-low interactions, low-high interactions, and high-high interactions respectively. To treat the high-low interactions, we use

$$|P_j((P_k f)(P_\ell g))| \lesssim (|P_k f| Mg) * k_j;$$

here k_j is the absolute value of the convolution kernel of P_j. To treat the low-high interactions, we use

$$|P_j((P_k f)(P_\ell g))| \lesssim (Mf |P_\ell g|) * k_j;$$

to treat the high-high interactions, we may use any of the two inequalities above. Now we need to estimate

$$\left\| \left(\sum_{j=1}^{\infty} |2^{\alpha j} P_j(fg)|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)}.$$

Triangle inequality, Question 16 of Homework 4, together with the above considerations allows one to bound this by

$$\left\| \left(\sum_{j=1}^{\infty} \left(2^{\alpha j} \sum_{k \simeq j} |P_k f| Mg \right)^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} + \left\| \left(\sum_{j=1}^{\infty} \left(2^{\alpha j} \sum_{\ell \simeq j} Mf |P_\ell g| \right)^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)}$$
$$+ \left\| \left(\sum_{j=1}^{\infty} \left(2^{\alpha j} \sum_{k >> j} |P_k f| Mg \right)^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)}$$

The first two terms can be estimated by bounding Mg and Mf by their L^∞ norms, which are in turn bounded by their $W^{\alpha,p}$ norms. The last term can be bounded by using triangle inequality:

$$\sum_{r=1}^{\infty} \left\| \left(\sum_{j=1}^{\infty} \left(2^{\alpha j} |P_{j+r} f| Mg \right)^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)}$$

which upon shifting summation and bounding Mg by L^∞ is bounded by

$$\sum_{r=1}^{\infty} 2^{-\alpha r} \left\| \left(\sum_{j=r}^{\infty} \left(2^{\alpha j} |P_j f| \right)^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \|g\|_{W^{\alpha,p}(\mathbb{R}^n)} \lesssim \|f\|_{W^{\alpha,p}(\mathbb{R}^n)} \|g\|_{W^{\alpha,p}(\mathbb{R}^n)}.$$

This almost finishes the proof of the desired estimate; one still needs to estimate $\|P_0(fg)\|_{L^p(\mathbb{R}^n)}$, but that is much easier.)