# MATH6081A Homework 6

1. (a) Suppose $m \in \mathbb{R}$, $\gamma, \delta \in [0,1]$, and $a \in S^m_{\gamma,\delta}$. Show that $T_a$ maps $\mathcal{S}(\mathbb{R}^n)$ into itself, and that the map $T_a \colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous.

   (b) Suppose $m \in \mathbb{R}$, $\gamma \in [0,1]$, $\delta \in [0,1)$, and $c \in CS^m_{\gamma,\delta}$. Show that for every $\varepsilon > 0$, $T_{[c],\varepsilon}$ maps $\mathcal{S}(\mathbb{R}^n)$ into itself, and that $T_{[c],\varepsilon} f$ converges in the topology of $\mathcal{S}(\mathbb{R}^n)$ as $\varepsilon \to 0^+$ for every $f \in \mathcal{S}(\mathbb{R}^n)$. Also show that the map $T_{[c]} \colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous.

2. Suppose $c \in CS^m$ for some $m \in \mathbb{R}$. Show that there exists $a \in S^m$ such that

$$T_{[c]} f = T_a f$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$. Also show that

$$a(x,\xi) \sim \sum_{\gamma} \frac{(2\pi i)^{-|\gamma|}}{\gamma!} \partial_y^\gamma \partial_\xi^\gamma \, c(x,y,\xi)|_{y=x},$$

in the sense that

$$a(x,\xi) - \sum_{|\gamma|<N} \frac{(2\pi i)^{-|\gamma|}}{\gamma!} \partial_y^\gamma \partial_\xi^\gamma \, c(x,y,\xi)|_{y=x} \in S^{m-N}$$

for all $N \in \mathbb{N}$. (Hint: Let $\eta$ be a fixed $C^\infty$ function on $\mathbb{R}^n$ supported on $B(0,2)$, such that $\eta \equiv 1$ on $B(0,1)$. Then for any $\varepsilon > 0$ and any $f \in \mathcal{S}(\mathbb{R}^n)$, we have

$$T_{[c]} f(x) = \lim_{\varepsilon \to 0^+} T_{[c_{\varepsilon,1}]} f(x) + \lim_{\varepsilon \to 0^+} T_{[c_{\varepsilon,2}]} f(x)$$

where

$$c_{\varepsilon,1}(x,y,\xi) := c(x,y,\xi)\eta(x-y)\eta(\varepsilon\xi),$$
$$c_{\varepsilon,2}(x,y,\xi) := c(x,y,\xi)(1-\eta)(x-y)\eta(\varepsilon y)\eta(\varepsilon\xi)$$

and the convergence is in $\mathcal{S}(\mathbb{R}^n)$ (actually we just need pointwise convergence here). Now

$$T_{[c_{\varepsilon,1}]} f(x) = \int_{\mathbb{R}^n} a_{\varepsilon,1}(x,\xi) \widehat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi,$$

where

$$\begin{aligned} a_{\varepsilon,1}(x,\xi) &:= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} c_{\varepsilon,1}(x,y,\zeta) e^{2\pi i (x-y)\cdot(\zeta-\xi)} dy d\zeta \\ &= \int_{\mathbb{R}^n} \widehat{c_{\varepsilon,1}}(x,\zeta-\xi,\zeta)^{2\pi i x\cdot(\zeta-\xi)} d\zeta \\ &= \int_{\mathbb{R}^n} \widehat{c_{\varepsilon,1}}(x,\zeta,\xi+\zeta)^{2\pi i x\cdot\zeta} d\zeta; \end{aligned}$$

here $\widehat{c}_{\varepsilon,1}$ is the Fourier transform of $c$ in the middle variable. Taylor expanding, we get, for $N \geq m$, that

$$\widehat{c_{\varepsilon,1}}(x,\zeta,\xi+\zeta) = \sum_{|\gamma|<N} \frac{1}{\gamma!} \partial_\xi^\gamma \widehat{c_{\varepsilon,1}}(x,\zeta,\xi)\zeta^\gamma + R_N(x,\zeta,\xi),$$

where the remainder $R_N(x,\zeta,\xi)$ satisfies, uniformly in $\varepsilon$, that

$$\sup_{x\in\mathbb{R}^n}\sup_{\xi\in\mathbb{R}^n}|\partial_x^\alpha\partial_\xi^\beta R_N(x,\zeta,\xi)| \lesssim_{N,\bar{N},\alpha,\beta} |\zeta|^N(1+|\zeta|)^{-\bar{N}}$$

for all $\zeta\in\mathbb{R}^n$ and all $\bar{N}\in\mathbb{N}$, and

$$\sup_{x\in\mathbb{R}^n}|\partial_x^\alpha\partial_\xi^\beta R_N(x,\zeta,\xi)| \lesssim_{N,\bar{N},\alpha,\beta} |\zeta|^N(1+|\zeta|)^{-\bar{N}}(1+|\xi|)^{m-N-|\beta|}$$

whenever $|\xi|>2|\zeta|$ and $\bar{N}\in\mathbb{N}$. Plugging this back into the formula for $a_{\varepsilon,1}$, we get

$$a_{\varepsilon,1}(x,\xi)=\sum_{|\gamma|<N}\frac{(2\pi i)^{-|\gamma|}}{\gamma!}\partial_y^\gamma\partial_\xi^\gamma\, c_{\varepsilon,1}(x,y,\xi)|_{y=x}+e_{\varepsilon,1}(x,\xi)$$

where $e_{\varepsilon,1}(x,\xi)\in S^{m-N}$ uniformly in $\varepsilon$. Next,

$$T_{[c_{\varepsilon,2}]}f(x)=\int_{\mathbb{R}^n}a_{\varepsilon,2}(x,\xi)\widehat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi,$$

where

$$a_{\varepsilon,2}(x,\xi):=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}c_{\varepsilon,2}(x,y,\zeta)e^{2\pi i(x-y)\cdot(\zeta-\xi)}dyd\zeta.$$

But since $c_{\varepsilon,2}(x,y,\zeta)$ is supported away from $x=y$, one can integrate by parts using

$$e^{2\pi i(x-y)\cdot(\zeta-\xi)}=\frac{\Delta_\zeta e^{2\pi i(x-y)\cdot(\zeta-\xi)}}{-4\pi^2|x-y|^2}$$

to gain decay in $\zeta$. Also, since $c_{\varepsilon,2}(x,y,\zeta)$ is compactly supported in $y$, one can integrate by parts using

$$e^{2\pi i(x-y)\cdot(\zeta-\xi)}=\frac{(I-\Delta_y)e^{2\pi i(x-y)\cdot(\zeta-\xi)}}{1+4\pi^2|\xi-\zeta|^2}$$

to gain decay in $|\xi-\zeta|$. This shows

$$|a_{\varepsilon,2}(x,\xi)|\lesssim\int_{\mathbb{R}^n}\int_{|y-x|\geq 1}\frac{(1+|\zeta|)^{m-N_1}}{|x-y|^{N_1}(1+|\xi-\zeta|)^{N_2}}dyd\zeta$$

which is $\lesssim_{\bar{N}}(1+|\xi|)^{-\bar{N}}$ for all $\bar{N}\in\mathbb{N}$ upon dividing the domain of integration into where $|\zeta|\leq|\xi|/2$ and $|\zeta|>|\xi|/2$, and the estimate is uniform in $\varepsilon$. Similarly, $|\partial_x^\alpha\partial_\xi^\beta a_{\varepsilon,2}(x,\xi)|\lesssim_{\bar{N},\alpha,\beta}(1+|\xi|)^{-\bar{N}}$ for all $\bar{N}\in\mathbb{N}$, uniformly in $\varepsilon$. Thus $a_{\varepsilon,2}(x,\xi)\in S^{m-N}$ for all $N\in\mathbb{N}$, uniformly in $\varepsilon$. It remains to observe that

$$\lim_{\varepsilon\to 0^+}\partial_y^\gamma\partial_\xi^\gamma\, c_{\varepsilon,1}(x,y,\xi)|_{y=x}=\partial_y^\gamma\partial_\xi^\gamma\, c(x,y,\xi)|_{y=x}$$

pointwisely for all $|\gamma|<N$, and that there exists a symbol $e_N(x,\xi)\in S^{m-N}$, such that

$$\lim_{\varepsilon\to 0^+}(e_{\varepsilon,1}(x,\xi)+a_{\varepsilon,2}(x,\xi))=e_N(x,\xi)$$

pointwisely. By dominated convergence, this shows that if

$$a(x,\xi) := \sum_{|\gamma|<N} \frac{(2\pi i)^{-|\gamma|}}{\gamma!} \partial_y^\gamma \partial_\xi^\gamma \, c(x,y,\xi)|_{y=x} + e_N(x,\xi),$$

then

$$T_{[c]} f(x) = \int_{\mathbb{R}^n} a(x,\xi) \widehat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi = T_a f(x)$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$; note that $a(x,\xi)$ satisfies the desired asymptotic expansion by construction.)

3. (a) Show that if $a \in S^m$, then the formal adjoint of $T_a$ is given by a pseudodifferential operator with symbol $a^*$, where

$$a^*(x,\xi) \sim \sum_\gamma \frac{(2\pi i)^{-|\gamma|}}{\gamma!} \partial_y^\gamma \partial_\xi^\gamma \, \overline{a(y,\xi)}\Big|_{y=x}.$$

   (b) Show that if $a_1 \in S^{m_1}$ and $a_2 \in S^{m_2}$, then $T_{a_1} T_{a_2}$ is a pseudodifferential operator with symbol $a$, where

$$a(x,\xi) \sim \sum_\gamma \frac{(2\pi i)^{-|\gamma|}}{\gamma!} \partial_\xi^\gamma a_1(x,\xi) \partial_x^\gamma a_2(x,\xi).$$

   (Hint: Use the previous question. Note that $T_{a_1} T_{a_2} = T_{a_1} T^*_{a_2^*}$ has compound symbol $a_1(x,\xi)\overline{a_2^*(y,\xi)}$, and part (a) shows that

$$a_2(x,\xi) \sim \sum_\gamma \frac{(2\pi i)^{-|\gamma|}}{\gamma!} \partial_y^\gamma \partial_\xi^\gamma \, \overline{a_2^*(y,\xi)}\Big|_{y=x}.$$

   See also Stein's *Harmonic Analysis*, Chapter VI, Section 3 for a more direct proof.)

4. (a) Show that for any sequence of real numbers $c_0, c_1, \dots$, there exists a $C^\infty$ function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f^{(k)}(0) = c_k$ for all $k \geq 0$. (Hint: Consider

$$f(x) = \sum_{k=0}^\infty \eta\left(\frac{x}{\varepsilon_k}\right) c_k \frac{x^k}{k!}$$

   where $\eta \in C_c^\infty(\mathbb{R})$ and $\varepsilon_k$ is a sequence of positive numbers that tend to 0 very rapidly.)

   (b) Suppose $m \in \mathbb{R}$. Given a sequence of symbols $a_0, a_1, \dots$ with $a_k \in S^{m-k}$ for all $k \geq 0$, show that there exists a symbol $a \in S^m$, such that for every $N \in \mathbb{N}$, there exists $e_N \in S^{m-N}$ with

$$a(x,\xi) = \sum_{k=0}^{N-1} a_k(x,\xi) + e_N(x,\xi).$$

   (Hint: Mimic the proof of (a).)

5. (Hilbert-Schmidt norm) Let $T$ be an integral operator defined by

$$Tf(x) = \int K(x,y)f(y)dy,$$

where

$$\|K(x,y)\|_{L^2(dxdy)} \leq C.$$

Show that $T$ is bounded on $L^2$ with operator norm $\leq C$. (Hint: Estimate $|Tf(x)|^2$ using Cauchy-Schwarz:

$$|Tf(x)|^2 \leq \left(\int |K(x,y)|^2 dy\right)\left(\int |f(y)|^2 dy\right).$$

Then integrate in $x$.) Can you interpret this as a bound for the operator norms of matrices in linear algebra?

6. (Schur's test) Let $T$ be an integral operator defined by

$$Tf(x) = \int K(x,y)f(y)dy,$$

where

$$\sup_x \|K(x,y)\|_{L^1(dy)} \leq A \quad \text{and} \quad \sup_y \|K(x,y)\|_{L^1(dx)} \leq B.$$

Show that $T$ is bounded on $L^2$ with operator norm $\leq \sqrt{AB}$. (Hint: Estimate $|Tf(x)|^2$ using Cauchy-Schwarz:

$$|Tf(x)|^2 \leq A^2 \int |K(x,y)||f(y)|^2 dy.$$

Then integrate in $x$. Alternatively, use duality: Assume $\|f\|_{L^2} = \|g\|_{L^2} = 1$. Then

$$\begin{aligned}\int Tf(x)g(x)dx &\leq \int\int |K(x,y)||f(y)||g(x)|dydx \\ &\leq \frac{\sqrt{A}}{2\sqrt{B}}\int\int |K(x,y)||f(y)|^2 dydx + \frac{\sqrt{B}}{2\sqrt{A}}\int\int |K(x,y)||g(x)|^2 dydx\end{aligned}$$

where we used AM-GM in the last inequality. Yet another way is to interpolate between $T\colon L^1 \to L^1$ and $T\colon L^\infty \to L^\infty$.) Can you interpret this as a bound for the operator norms of matrices?

7. Let $\{\gamma_j\}_{j\in\mathbb{Z}}$ be an $\ell^1$ sequence of non-negative real numbers, with $A := \sum_{j\in\mathbb{Z}} \gamma_j$.

   (a) Prove that if $\{t_j\}_{j=1}^J$ is a finite sequence of real numbers with $|t_i t_j| \leq \gamma_{i-j}^2$ for all $i, j = 1, \ldots, J$, then $\sum_{j=1}^J |t_j| \leq A$. (Hint: Let $j_0$ be the index where $|t_{j_0}| = \max_j |t_j|$. Then

$$\sum_{j=1}^J |t_j| \leq |t_{j_0}|^{1/2} \sum_{j=1}^J |t_j|^{1/2} \leq |t_{j_0}|^{1/2} \sum_{j=1}^J |t_{j_0}|^{-1/2} \gamma_{j-j_0}$$

   which is bounded by $A$.)

(b) Prove the original version of Cotlar's lemma: If $\{T_j\}_{j=1}^J$ is a finite sequence of commuting self-adjoint operators on a Hilbert space $H$, with

$$\|T_i T_j\|_{H\to H} \leq \gamma_{i-j}^2$$

for all $i, j = 1, \dots, J$, then for $T = \sum_{j=1}^J T_j$, we have $\|T\|_{H\to H} \leq A$. (Hint: Simultaneously diagonalize the $T_j$'s, and use part (a).)

8. Let $\{f_j\}$ be a sequence in a Hilbert space $H$, and $A \in \mathbb{R}$. Suppose for any sequence $\{\varepsilon_j\}$ that satisfies $\varepsilon_j \in \{-1, 0, 1\}$ for all $j$ and has only finitely many non-zero terms, we have

$$\left\| \sum_j \varepsilon_j f_j \right\| \leq A.$$

Show that $\sum_j f_j$ converges in $H$. (Hint: Suppose not, then there exists some $\delta > 0$, and some sequences $\{M_k\}$, $\{N_k\}$ with $M_k < N_k < M_{k+1}$ for all $k \in \mathbb{N}$, such that if

$$F_k = \sum_{M_k \leq |j| \leq N_k} f_j,$$

then $\|F_k\| \geq \delta$. As a result, $\sum_{k=1}^K \|F_k\|^2 > A^2$ if $K$ is sufficiently large. But our assumptions imply that

$$\sum_{k=1}^K \|F_k\|^2 \leq A^2$$

for all $K \in \mathbb{N}$. Hence the desired contradition.)

9. Suppose $\{T_j\}_{j\in\mathbb{Z}}$ is a family of bounded linear operators on a Hilbert space $H$, for which there exists a non-negative $\ell^1$ sequence $\{\gamma_j\}_{j\in\mathbb{Z}}$ with $\sum_j \gamma_j = A < \infty$ such that

$$\|T_j\|_{H\to H} \leq A \quad \text{for all } j \in \mathbb{Z}$$
$$\|T_k^* T_j\|_{H\to H} \leq \gamma_j \gamma_k \quad \text{for all } j \neq k$$
$$T_j T_k^* = 0 \quad \text{for all } j \neq k.$$

Show that $\sum_{j\in\mathbb{Z}} T_j$ converges strongly to a bounded linear operator $T$ on $H$, with

$$\|T\|_{H\to H} \leq \sqrt{2} A.$$

This is a cruder version of Cotlar-Stein, which can sometimes be used as a substitute in applications and admits a much more direct proof. (Hint: The ranges of $T_j^*$'s are orthogonal, so there exists mutually orthogonal projections $E_j$'s such that $T_j^* = E_j T_j^*$ for all $j \in \mathbb{Z}$. This shows $T_j = T_j E_j$ for all $j \in \mathbb{Z}$, so

$$\begin{aligned}\left\| \sum_{|j|\leq N} T_j f \right\|^2 &= \sum_{|j|\leq N} \|T_j f\|^2 + \sum_{\substack{|j|,|k|\leq N \\ j\neq k}} \langle T_k^* T_j f, f\rangle \\ &\leq \sum_{|j|\leq N} A^2 \|E_j f\|^2 + \sum_{\substack{|j|,|k|\leq N \\ j\neq k}} \gamma_j \gamma_k \|f\|^2 \leq 2A^2 \|f\|^2\end{aligned}$$

uniformly for $N \in \mathbb{N}$. This also holds with $T_j f$ replaced by $\pm T_j f$. One can thus let $N \to \infty$ by invoking the result in the previous question.)

10. Let $H$ be the Hilbert transform on $\mathbb{R}$, defined as convolution with the tempered distribution given by the principal value of $\frac{1}{\pi x}$. The goal of this question is to prove the $L^2$ boundedness of $H$ by appealing to Cotlar-Stein.

    Let $\psi(x)$ be a smooth even function on $\mathbb{R}$, for which $\psi(x) = 1$ if $|x| \leq 1$, and $\psi(x) = 0$ if $|x| \geq 2$. Let $\varphi(x) = \psi(x) - \psi(2x)$ so that $\varphi$ is supported on $\{1/2 \leq |x| \leq 2\}$, and $\sum_{j \in \mathbb{Z}} \varphi(2^j x) = 1$ for all $x \neq 0$. For each $j \in \mathbb{Z}$, let

$$k_j(x) = \frac{1}{\pi} \frac{\varphi(2^j x)}{x}.$$

    Also let

$$T_j f(x) = f * k_j(x) \quad \text{for } f \in \mathcal{S}(\mathbb{R}).$$

    (a) Show that $\sum_{|j| \leq N} k_j$ converges to the principal value of $\frac{1}{\pi x}$ in the topology of $\mathcal{S}'(\mathbb{R})$ when $N \to +\infty$. Hence for $f \in \mathcal{S}(\mathbb{R}^n)$, we have pointwise convergence of $\sum_{|j| \leq N} T_j f(x)$ to $Hf(x)$ as $N \to \infty$.

    (b) Show that

$$|k_j(x)| \lesssim \frac{2^j}{(1 + 2^j |x|)^2}$$

    and more generally

$$|\partial_x^\alpha k_j(x)| \lesssim \frac{2^{j(1+|\alpha|)}}{(1 + 2^j |x|)^2}.$$

    Also show that

$$\int_{\mathbb{R}^n} k_j(x) dx = 0.$$

    (c) Using Cotlar-Stein, show that $\sum_{|j| \leq N} T_j$ converges strongly to $H$ in $L^2$, and conclude that $H$ is bounded on $L^2(\mathbb{R})$.

11. (a) (Calderón-Vaillancourt theorem) Let $a \in S^0_{0,0}$. Show that the pseudodifferential operator $T_a$, initially defined on $\mathcal{S}(\mathbb{R}^n)$, extends to a bounded linear operator on $L^2(\mathbb{R}^n)$. (Hint: Let $S = T_a \mathcal{F}^{-1}$ where $\mathcal{F}^{-1}$ is the inverse Fourier transform on $\mathbb{R}^n$. Then by Plancherel, it suffices to show that $S$ extends to a bounded linear operator on $L^2(\mathbb{R}^n)$. Let $\sum_{j_1 \in \mathbb{Z}^n} \phi(x - j_1) = 1$ be a smooth partition of unity on $\mathbb{R}^n$, where $\phi$ is a non-negative smooth function with compact support on $B(0,1)$. Similarly, $\sum_{j_2 \in \mathbb{Z}^n} \phi(\xi - j_2) = 1$ is a smooth partition of unity on the $\xi$ space. For $j = (j_1, j_2) \in \mathbb{Z}^n \times \mathbb{Z}^n$, let

$$S_j f(x) = \int_{\mathbb{R}^n} \phi(x - j_1) a(x, \xi) \phi(\xi - j_2) f(\xi) e^{2\pi i x \cdot \xi} d\xi$$

    for $f \in \mathcal{S}(\mathbb{R}^n)$, so that $Sf = \sum_{j \in \mathbb{Z}^{2n}} S_j f$ for all $f \in \mathcal{S}(\mathbb{R}^n)$ with convergence in $\mathcal{S}(\mathbb{R}^n)$. Now for all $j, k \in \mathbb{Z}^{2n}$ and $f \in \mathcal{S}(\mathbb{R}^n)$, we have

$$S_j S_k^* f(x) = \int_{\mathbb{R}^n} K_{j,k}(x, y) f(y) dy$$

    where

$$K_{j,k}(x, y) := \int_{\mathbb{R}^n} \phi(x - j_1) a(x, \xi) \phi(\xi - j_2) \phi(\xi - k_2) \overline{a(y, \xi)} \phi(y - k_1) e^{2\pi i (x-y) \cdot \xi} d\xi.$$

But $K_{j,k}(x,y)=0$ if $|j_2-k_2|\geq 2$,

$$|K_{j,k}(x,y)|\lesssim \phi(x-j_1)\phi(y-k_1),$$

and if one integrates by parts using

$$e^{2\pi i(x-y)\cdot\xi}=\frac{\Delta_\xi e^{2\pi i(x-y)\cdot\xi}}{-4\pi^2|x-y|^2},$$

one gets

$$|K_{j,k}(x,y)|\lesssim_N \frac{1}{|j_1-k_1|^N}\phi(x-j_1)\phi(y-k_1)$$

for any $N\in\mathbb{N}$, if $|j_1-k_1|\geq 3$. Thus

$$\|S_jS_k^*\|_{L^2\to L^2}\lesssim \frac{1}{(1+|j-k|)^N}.$$

By symmetry, the same is true for $\|S_k^*S_j\|_{L^2\to L^2}$. One can then apply Cotlar-Stein. Note that this gives another proof that $T_a\colon L^2\to L^2$ if $a\in S^0=S^0_{1,0}$.)

(b) More generally, let $a\in S^0_{\gamma,\gamma}$ for some $\gamma\in[0,1)$. Show that the pseudodifferential operator $T_a$, initially defined on $\mathcal{S}(\mathbb{R}^n)$, extends to a bounded linear operator on $L^2(\mathbb{R}^n)$. (Hint: Let $a\in S^0_{\gamma,\gamma}$ with $\gamma\in[0,1)$. By replacing $a(x,\xi)$ by $a_\varepsilon(x,\xi):=a(x,\xi)\eta(\varepsilon x)\eta(\varepsilon\xi)$ for some $\eta\in C^\infty_c(\mathbb{R}^n)$ with $\eta(0)=1$, we may assume that $a(x,\xi)$ has compact support in both $x$ and $\xi$; this works because $a_\varepsilon(x,\xi)\in S^0_{\gamma,\gamma}$ uniformly in $\varepsilon$, and $T_{a_\varepsilon}f\to T_af$ in $\mathcal{S}(\mathbb{R}^n)$ for all $f\in\mathcal{S}(\mathbb{R}^n)$ as $\varepsilon\to 0^+$. Define Littlewood-Paley decomposition $I=P_0+\sum_{j=1}^\infty \Delta_j$ by letting

$$P_0f=\mathcal{F}^{-1}[\psi(\xi)\widehat{f}(\xi)],$$

$$\Delta_jf=\mathcal{F}^{-1}[\varphi(2^{-j}\xi)\widehat{f}(\xi)] \quad \text{for } j\geq 1,$$

where $\psi(\xi)$ is a real-valued smooth function with compact support on the unit ball $B(0,2)$, with $\psi(\xi)\equiv 1$ on $B(0,1)$, and $\varphi(\xi)=\psi(\xi)-\psi(2\xi)$ so that $\varphi$ is supported on the annulus $\{1/2\leq|\xi|\leq 2\}$. Write

$$T_a=T_aP_0+\sum_{j=1}^\infty T_a\Delta_j.$$

$T_aP_0$ clearly extends to a bounded linear operator on $L^2(\mathbb{R}^n)$, since its symbol is in $S^0_{1,0}$. We claim that $\sum_{j \text{ odd}} T_a\Delta_j$ extends to a bounded linear operator on $L^2(\mathbb{R}^n)$: indeed if $j,k$ are odd and distinct, then $T_a\Delta_j(T_a\Delta_k)^*=0$, and

$$(T_a\Delta_k)^*(T_a\Delta_j)f(x)=\int_{\mathbb{R}^n}K_{j,k}(x,y)f(y)dy$$

for all $f\in\mathcal{S}(\mathbb{R}^n)$, where

$$K_{j,k}(x,y):=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\overline{a(z,\eta)}\varphi(2^{-k}\eta)a(z,\eta)\varphi(2^{-k}\xi)e^{2\pi i((z-y)\cdot\xi+(x-z)\cdot\eta)}dzd\eta d\xi$$

satisfies

$$|K_{j,k}(x,y)| \lesssim_N 2^{\max\{j,k\}(\gamma-1)N} \int_{\mathbb{R}^n} (1+|x-z|)^{-N}(1+|z-y|)^{-N} dz.$$

(The interchange of order of integration is justified by Fubini, since $a(x,\xi)$ has compact $x$ and $\xi$ supports.) If $N > n$, then Schur's test shows that $\|(T_a \Delta_k)^*(T_a \Delta_j)\|_{L^2 \to L^2} \lesssim_N 2^{\max\{j,k\}(\gamma-1)N}$, so by Cotlar-Stein (or the result of Question 9), to show that $\sum_{j \text{ odd}} T_a \Delta_j$ extends to a bounded linear operator on $L^2(\mathbb{R}^n)$, it remains to show that $\|T_a \Delta_j\|_{L^2 \to L^2}$ are uniformly bounded in $j$. This follows from part (a), since if $D_j$ is the $L^2$-norm-preserving dilation given by

$$D_j f(x) := 2^{j\gamma n/2} f(2^{j\gamma} x),$$

then

$$\|T_a \Delta_j\|_{L^2 \to L^2} = \|D_j^{-1} T_a \Delta_j D_j\|_{L^2 \to L^2},$$

whereas $D_j^{-1} T_a \Delta_j D_j$ is a pseudodifferential operator with symbol

$$a(2^{-j\gamma} x, 2^{j\gamma} \xi) \varphi(2^{-j} 2^{j\gamma} \xi),$$

which is in $S^0_{0,0}$ uniformly in $j$ since $a \in S^0_{\gamma,\gamma}$. Similarly $\sum_{j \text{ even}} T_a \Delta_j$ extends to a bounded linear operator on $L^2(\mathbb{R}^n)$.)