MATH6081A Homework 7

1. (a) Let f be a BMO function on $\mathbb{R}^n$. Suppose for every $\tau \in \mathbb{R}^n$, there exists a constant c_τ such that

$$f(x+\tau) = f(x) + c_\tau \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Show that f is constant, and hence f is identified with 0 in the BMO space. (Hint: First show that $c_{\tau_1+\tau_2} = c_{\tau_1} + c_{\tau_2}$ for all $\tau_1, \tau_2 \in \mathbb{R}^n$, and that c_τ is continuous as a function of τ by noting that

$$c_\tau = \fint_{B(0,1)} f(x+\tau)dx - \fint_{B(0,1)} f(x)dx$$

for all $\tau \in \mathbb{R}^n$. Hence c_τ is a linear function of τ, which shows $f(x)$ is equal to an affine function of x a.e. on $\mathbb{R}^n$. The only affine functions that are in BMO are constants, by considering averages of $|f(x) - f_{B(0,R)}|$ over $B(0,R)$ as $R \to +\infty$.)

(b) Suppose $K \in \mathcal{S}'(\mathbb{R}^n)$ is a tempered distribution with $\widehat{K} \in L^\infty(\mathbb{R}^n)$. Suppose also K agrees with a measurable function K_0 away from the origin, for which

$$\sup_{y \neq 0} \int_{|x| \geq 2|y|} |K_0(x-y) - K_0(x)| dx < \infty.$$

Let $T\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be given by $Tf = f * K$ for all $f \in \mathcal{S}(\mathbb{R}^n)$. We knew already that T extends as a continuous linear operator from $L^\infty(\mathbb{R}^n)$ to BMO; if $g \in L^\infty(\mathbb{R}^n)$, we write Tg for a globally defined BMO function on $\mathbb{R}^n$ with some fixed normalization (say $\int_{B(0,1)} Tg(x)dx = 0$).

(i) Show that T commutes with translation, in the sense that if $g \in L^\infty(\mathbb{R}^n)$ and $g_\tau(x) := g(x+\tau)$, then for every $\tau \in \mathbb{R}^n$, there exists a constant c_τ such that

$$Tg_\tau(x) = Tg(x+\tau) + c_\tau \quad \text{for a.e. } x \in \mathbb{R}^n.$$

(ii) Show that $T(1) = 0$ as a function in BMO. (Hint: Apply part (a) to $f := T(1)$.)

2. A function a on $\mathbb{R}^n$ is said to be an Hardy $\mathcal{H}^1$ atom (associated to a ball B) if a is measurable, supported on B, $\|a\|_{L^2(B)} \leq |B|^{-1/2}$, and $\int_B a(x)dx = 0$. A function f on $\mathbb{R}^n$ is in Hardy $\mathcal{H}^1$, if there exists a sequence $a_1, a_2, \dots$ of Hardy $\mathcal{H}^1$ atoms and a complex sequence $\lambda_1, \lambda_2, \dots$ with $\sum_{k=1}^\infty |\lambda_k| < \infty$ such that

$$f = \sum_{k=1}^\infty \lambda_k a_k.$$

For f in Hardy $\mathcal{H}^1$, let $\|f\|_{\mathcal{H}^1}$ be the infimum of $\sum_{k=1}^\infty |\lambda_k|$, over all possible decompositions of f into $\sum_{k=1}^\infty \lambda_k a_k$, where $a_1, a_2, \dots$ are Hardy $\mathcal{H}^1$ atoms.

(a) Show that Hardy $\mathcal{H}^1$ is a vector space.

(b) Show that $\|f\|_{\mathcal{H}^1}$ defines a norm on Hardy $\mathcal{H}^1$, and that $\mathcal{H}^1$ embeds continuously into L^1. (Hint: If f is in Hardy $\mathcal{H}^1$ then $f \in L^1$ with $\|f\|_{L^1} \leq \|f\|_{\mathcal{H}^1}$. Hence if $\|f\|_{\mathcal{H}^1} = 0$ then $f = 0$ a.e. Now check the triangle inequality.)

(c) Show that Hardy $\mathcal{H}^1$ is complete. (Hint: Suppose $\{f_k\}_{k=1}^\infty$ is a sequence in Hardy $\mathcal{H}^1$ with $\|f_k\|_{\mathcal{H}^1} \leq 2^{-k}$ for all $k \geq 1$. It suffices to show that $\sum_{k=1}^\infty f_k$ is in Hardy $\mathcal{H}^1$ (why?). But each f_k admits a decomposition into sums of atoms, hence so does $\sum_{k=1}^\infty f_k$. This completes the proof.)

(d) Show that the dual space of Hardy $\mathcal{H}^1$ on $\mathbb{R}^n$ is $BMO(\mathbb{R}^n)$. (Hint: Let $\mathcal{H}^1_a$ be the subspace of Hardy $\mathcal{H}^1$, that consists of finite linear combinations of Hardy $\mathcal{H}^1$ atoms. Then $\mathcal{H}^1_a$ is a dense subspace of Hardy $\mathcal{H}^1$. Given $g \in BMO(\mathbb{R}^n)$, define

$$\langle f, g \rangle = \int_{\mathbb{R}^n} f(x)\overline{g(x)}dx$$

for all $f \in \mathcal{H}^1_a(\mathbb{R}^n)$. Then

$$\langle f, g \rangle = \lim_{N\to\infty} \int_{\mathbb{R}^n} f(x)\overline{g_N(x)}dx$$

for all $f \in \mathcal{H}^1_a(\mathbb{R}^n)$, where g_N is the truncation of g, given by

$$g_N(x) = \begin{cases} N & \text{if } g(x) \geq N \\ g(x) & \text{if } -N \leq g(x) \leq N \\ -N & \text{if } g(x) \leq -N. \end{cases}$$

This limit exists by dominated convergence theorem, since $|g_N|$ is dominated by $|g|$, which is locally L^2. Now for $f \in \mathcal{H}^1_a$ and $\varepsilon > 0$, write

$$f = \sum_{k=1} \lambda_k a_k$$

where $a_1, a_2, \dots$ are Hardy $\mathcal{H}^1$ atoms and

$$\|f\|_{\mathcal{H}^1} \leq \sum_{k=1}^\infty |\lambda_k| + \varepsilon.$$

Then for every $N > 0$,

$$\int_{\mathbb{R}^n} f(x)\overline{g_N(x)}dx = \sum_{k=1}^\infty \lambda_k \int_{\mathbb{R}^n} a_k(x)\overline{g_N(x)}dx = \sum_{k=1}^\infty \lambda_k \int_{\mathbb{R}^n} a_k(x)(\overline{g_N(x) - c_{k,N}})dx,$$

where $c_{k,N} = \fint_{B_k} g_N$ and B_k is the ball associated to a_k, so

$$\begin{aligned} \left| \int_{\mathbb{R}^n} f(x)\overline{g_N(x)}dx \right| &\leq \sum_{k=1}^\infty |\lambda_k| \left(\frac{1}{|B_k|} \int_{B_k} |g_N(x) - c_{k,N}|^2 dx \right)^{1/2} \\ &\lesssim (\|f\|_{\mathcal{H}^1} + \varepsilon) \|g\|_{BMO} \end{aligned}$$

by the John-Nirenberg inequality and the fact that $\|g_N\|_{BMO} \lesssim \|g\|_{BMO}$ uniformly in N. Thus letting $N \to \infty$ and $\varepsilon \to 0^+$, we get

$$|\langle f, g\rangle| \lesssim \|f\|_{\mathcal{H}^1}\|g\|_{BMO}$$

for all $f \in \mathcal{H}^1_a$, and by density this shows every $g \in BMO$ defines a bounded linear functional on Hardy $\mathcal{H}^1$.

For the converse direction, suppose L is a linear functional on Hardy $\mathcal{H}^1$ with norm 1. For every ball $B \subset \mathbb{R}^n$, let $L^2_0(B)$ be the space of L^2 functions f on B with $\int_B f(x)dx = 0$, and equip $L^2_0(B)$ with the standard L^2 norm $\|f\|_{L^2(B)}$ so that $L^2_0(B)$ becomes a Hilbert space. Then L induces a bounded linear functional on $L^2_0(B)$ with norm $\lesssim |B|^{1/2}$, so there exists $g^{(B)} \in L^2_0(B)$ with $\|g^{(B)}\|_{L^2(B)} \lesssim |B|^{1/2}$ such that

$$L(f) = \int_B f(x)\overline{g^{(B)}(x)}dx$$

for all $f \in L^2_0(B)$. If $B_1 \cap B_2 \neq \emptyset$, then $g^{(B_1)} - g^{(B_2)}$ is a constant on $B_1 \cap B_2$. Thus one can define a global function g on $\mathbb{R}^n$, such that for every ball $B \subset \mathbb{R}^n$, there exists a constant c_B such that $g = g^{(B)} + c_B$. Now

$$\left(\sup_B \fint_B |g(x) - c_B|^2 dx\right)^{1/2} = \sup_B |B|^{-1/2}\|g^{(B)}\|_{L^2(B)} \lesssim 1.$$

Thus $g \in BMO(\mathbb{R}^n)$, and it is easy to check that $L(f) = \langle f, g\rangle$ for every $f \in \mathcal{H}^1_a$, as desired.)

3. Let $Tf = f * K$ be the singular integral operator as in Question 1(b).

(a) Show that T extends as a continuous linear operator from Hardy $\mathcal{H}^1$ to L^1. (Hint: It suffices to check this on atoms. Let a be an Hardy $\mathcal{H}^1$ atom associated to a ball B. Let B^* be the ball with the same center as B but twice the radius. Note that $\|Ta\|_{L^1(B^*)} \lesssim |B|^{1/2}\|Ta\|_{L^2}$, which can then be estimated by using L^2 theory. On the other hand, if $x \notin B^*$, then

$$Ta(x) = \int_{y \notin B^*} [K_0(x-y) - K_0(x-y_0)]a(y)dy$$

where y_0 is the center of B. One can then estimate $\|Ta\|_{L^1((B^*)^c)}$ using the estimates for the derivative of K_0.)

(b) By symmetry, part (a) also shows that the formal adjoint T^* of T extends as a continuous linear operator from Hardy $\mathcal{H}^1$ to L^1. In the lecture we proved that T extends as a continuous linear operator from L^∞ into BMO. Show that under these extensions,

$$\int_{\mathbb{R}^n} Tf(x)\overline{g(x)}dx = \int_{\mathbb{R}^n} f(x)\overline{T^*g(x)}dx$$

for all $f \in L^\infty$, $g \in \mathcal{H}^1_a$.

(c) Let a be an Hardy $\mathcal{H}^1$ atom. We saw that $T^*a \in L^1$. Show that

$$\int_{\mathbb{R}^n} T^*a(x)dx = 0.$$

This gives an alternative proof of the result in Question 1(b), namely $T(1) = 0$. (Hint: Since $T^*a \in L^1$, the Fourier transform of $T^*(a)$ is a continuous function, whose value at the origin is $\lim_{\xi \to 0} \widehat{K}(-\xi)\widehat{a}(\xi) = 0$. This gives the desired conclusion.)

4. Let T be as in the previous question. Show that T extends as a continuous linear map from $BMO(\mathbb{R}^n)$ into itself, with

$$\|Tf\|_{BMO} \lesssim_n \|f\|_{BMO}$$

for all $f \in BMO$. (Hint: Let $f \in BMO(\mathbb{R}^n)$. By dilation and translation invariance, it suffices to show that there exists a constant c, such that

$$\int_{B(0,1)} |Tf - c|\, dx \lesssim \|f\|_{BMO}.$$

Split $f = f_1 + f_2 + f_3$ where

$$f_1 = (f - f_{B(0,2)})\chi_{B(0,2)}, \quad f_2 = (f - f_{B(0,2)})\chi_{B(0,2)^c}, \quad f_3 = f_{B(0,2)},$$

and follow the proof that singular integrals map L^∞ into BMO; in particular, use L^2 theory to bound Tf_1, kernel derivative estimates to bound Tf_2, and note that $Tf_3 = 0$.)

5. Suppose $a \in BMO(\mathbb{R}^n)$, $\Phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \Phi(x)dx = 0$. For $t > 0$, let $\Phi_t(x) = t^n\Phi(tx)$. Show that $a * \Phi_t \in L^\infty(\mathbb{R}^n)$ for all $t > 0$, with

$$\|a * \Phi_t\|_{L^\infty} \lesssim_{\Phi,n} \|a\|_{BMO} \quad \text{uniformly in } t > 0.$$

(Hint: Let $\|a\|_{BMO} = 1$. By dilation and translation invariance, it suffices to bound $|a * \Phi(0)|$ by a constant that depends only on Φ and n. But this follows since

$$|a * \Phi(0)| = \left| \int_{\mathbb{R}^n} (a(x) - a_{B(0,1)})\Phi(x)dx \right| \lesssim_{\Phi,n} \int_{\mathbb{R}^n} \frac{|a(x) - a_{B(0,1)}|}{(1+|x|)^{n+1}} dx$$

at which point we may invoke Question 8(d) from Homework 5.)

6. Let $K_0(x,y)$ be a measurable function on $\mathbb{R}^n \times \mathbb{R}^n$ that satisfies

$$|K_0(x,y)| \lesssim |x-y|^{-n} \quad \text{for all } x, y \in \mathbb{R}^n.$$

Suppose

$$K_0(x,y) = -K_0(y,x) \quad \text{for all } x, y \in \mathbb{R}^n.$$

For $f, g \in \mathcal{S}(\mathbb{R}^n)$, define

$$\langle Tf, g \rangle = \lim_{\varepsilon \to 0} \iint_{|x-y|>\varepsilon} K_0(x,y)f(y)g(x)dydx.$$

(a) Show that for $f, g \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\langle Tf, g\rangle = \frac{1}{2}\iint K_0(x,y)[f(y)g(x) - g(y)f(x)]dydx.$$

(b) Hence show that T defines a continuous linear map $T\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, and that T is weakly bounded. (Hint: If $f \in \mathcal{S}(\mathbb{R}^n)$ and $g_n \to 0$ in $\mathcal{S}(\mathbb{R}^n)$, evaluate $\langle Tf, g_n\rangle$ using (a), and estimate that by splitting the integral into two parts, one where $|x-y| \leq 1$, the other where $|x-y| \geq 1$. On the part where $|x-y| \leq 1$, write

$$f(y)g_n(x) - g_n(y)f(x) = [f(y) - f(x)]g_n(y) + f(x)[g_n(y) - g_n(x)]$$

and use that $\|\nabla f\|_{L^\infty}\|g_n\|_{L^1} + \|\nabla g_n\|_{L^\infty}\|f\|_{L^1} \to 0$ as $n \to \infty$. On the part of the integral where $|x-y| \geq 1$, bound $|K_0(x,y)| \lesssim 1$, and note that $\|f_n\|_{L^1}\|g\|_{L^1} \to 0$ as $n \to \infty$. This shows that T maps $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^n)$. Reversing the role of f and g in the above argument, we see that $T\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is continuous. Finally, the argument we used to treat the integral where $|x-y| \leq 1$ can be easily modified to show that T is weakly bounded.)

7. Let $p_1 \in (1,\infty)$, $p_2 \in (1,\infty]$, $p \in (1,\infty)$, and

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}.$$

Suppose $f \in L^{p_1}(\mathbb{R}^n)$, $g \in L^{p_2}(\mathbb{R}^n)$. For $j \in \mathbb{Z}$, let

$$\Delta_j f = \mathcal{F}^{-1}[\varphi(2^{-j}\xi)\widehat{f}(\xi)] \quad \text{and} \quad S_j f = \mathcal{F}^{-1}[\psi(2^{-j}\xi)\widehat{f}(\xi)]$$

where φ is smooth with compact support on the annulus $\{1/2 \leq |\xi| \leq 2\}$ and ψ is smooth with compact support on the ball $\{|\xi| \leq 2\}$. Show that the paraproduct $\sum_{j=-\infty}^{\infty} \Delta_j f \cdot S_{j-3} g$ converges in $L^p(\mathbb{R}^n)$, with

$$\left\| \sum_{j=-\infty}^{\infty} \Delta_j f \cdot S_{j-3} g \right\|_{L^p} \lesssim \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}.$$

(Hint: Since $\sum_{|j|\leq N} \Delta_j f \to f$ in L^{p_1}, it suffices to prove that

$$\left\| \sum_{|j|\leq N} \Delta_j f \cdot S_{j-3} g \right\|_{L^p} \lesssim \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}$$

uniformly in N. Note that the function on the left hand side is a priori in $L^p(\mathbb{R}^n)$. By Littlewood-Paley inequality and the fact that $\Delta_j f \cdot S_{j-3} g$ has frequency support on an annulus $\{2^{j-2} \leq |\xi| \leq 2^{j+2}\}$, it suffices to estimate

$$\left\| \left(\sum_{|k|\leq N+3} \left| \Delta_k \sum_{|j-k|\leq 3} \Delta_j f \cdot S_{j-3} g \right|^2 \right)^{1/2} \right\|_{L^p}.$$

But one can take the finite sum out of the $L^p \ell^2$ norm, drop the Δ_k, and then estimate $|S_{j-3} g|$ by Mg, the Hardy-Littlewood maximal function of g. Thus the above is bounded by

$$\left\| \left(\sum_{|k| \leq N+3} |\Delta_j f|^2 \right)^{1/2} Mg \right\|_{L^p},$$

which by Hölder's inequality, the boundedness of M on L^{p_2}, and the Littlewood-Paley inequality is bounded by $\|f\|_{L^{p_1}} \|g\|_{L^{p_2}}$.)