MATH6081A Homework 8

1. Prove the Hausdorff-Young inequality, namely
$$\left\|\widehat{f}\right\|_{L^{p'}} \lesssim \|f\|_{L^p} \quad \text{for all } f \in L^p(\mathbb{R}^n) \text{ and all } 1 \leq p \leq 2.$$
In addition, when $1 < p \leq 2$ the above inequality can be refined using Lorentz spaces:
$$\left\|\widehat{f}\right\|_{L^{p',p}} \lesssim \|f\|_{L^p} \quad \text{for all } f \in L^p(\mathbb{R}^n) \text{ and all } 1 < p \leq 2.$$

2. (a) In Homework 1 we gave a direct proof of Young's convolution inequality, namely
$$\|f * g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q} \quad \text{if } p, q, r \in [1, \infty] \text{ with } 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$
Can you now give an alternative proof using interpolation?

   (b) We now give a refinement of Young's convolution inequality in the scale of Lorentz spaces.

   (i) Suppose $1 < p < \infty$. Show that if $f \in L^{p,\infty}$ and $g \in L^{p',1}$, then
$$\|f * g\|_{L^\infty} \lesssim \|f\|_{L^{p,\infty}}\|g\|_{L^{p',1}}.$$
(Hint: Use Question 16 of Homework 3.)

   (ii) Hence show that
$$\|f * g\|_{L^{r,q}} \lesssim \|f\|_{L^{p,\infty}}\|g\|_{L^q} \quad \text{and} \quad \|f * g\|_{L^r} \lesssim \|f\|_{L^{p,\infty}}\|g\|_{L^{q,r}}$$
if $p, q, r \in (1, \infty)$ with $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. In particular, for the same range of p, q, r, we have
$$\|f * g\|_{L^r} \lesssim \|f\|_{L^{p,\infty}}\|g\|_{L^q}.$$
This proves the last theorem in Lecture 3.

3. In Question 6 of Homework 6 we proved a Schur's lemma. We mentioned that there is an interpolation proof. Can you carry that out now? More generally, suppose T is an integral operator defined by $Tf(x) = \int K(x, y)f(y)dy$, where
$$\sup_x \|K(x, y)\|_{L^1(dy)} \leq A \quad \text{and} \quad \sup_y \|K(x, y)\|_{L^1(dx)} \leq B.$$
Show that if $p \in [1, \infty]$, then T is bounded on L^p with norm $\leq A^{1/p'}B^{1/p}$.

4. Let $p \in (0, \infty)$, $r \in (0, \infty]$, and f be a measurable function. Show that
$$\||f\||_{L^{p,r}} \simeq \|2^k W_k^{1/p}\|_{\ell^r(\mathbb{Z})}$$
where $W_k := |\{2^{k-1} < |f| \leq 2^k\}|$. (Hint: Since $|\{|f| > 2^k\}| \leq \sum_{\ell \in \mathbb{N}} W_{k+\ell}$, when $r < \infty$, the essence is in showing that
$$\sum_{k \in \mathbb{Z}} 2^{kr} \left(\sum_{\ell \in \mathbb{N}} W_{k+\ell} \right)^{r/p} \lesssim 1$$

when $\sum_{k\in\mathbb{Z}} 2^{kr} W_k^{r/p} = 1$. If $r/p \geq 1$, use the triangle inequality for $\ell^{r/p}$, to bound this by

$$\left[\sum_{\ell\in\mathbb{N}} \left(\sum_{k\in\mathbb{Z}} 2^{kr} W_{k+\ell}^{r/p} \right)^{p/r} \right]^{r/p};$$

on the other hand, if $r/p \leq 1$, just observe that

$$\left(\sum_{\ell\in\mathbb{N}} W_{k+\ell} \right)^{r/p} \leq \sum_{\ell\in\mathbb{N}} W_{k+\ell}^{r/p}.$$

The case $r = \infty$ is easier.)

5. (a) Show that $L^{p,r_1} \subseteq L^{p,r_2}$ if $p \in (0,\infty)$ and $0 < r_1 < r_2 \leq \infty$.
   (b) Show that $L^{p_0,\infty} \cap L^{p_1,\infty} \subseteq L^{p,r}$ if $0 < p_0 < p < p_0 \leq \infty$ and $r \in (0,\infty)$.

6. Let (X,μ) be a measure space. In Question 16 of Homework 3, we effectively showed that if $p \in (1,\infty)$, then

$$|||f|||_{L^{p,\infty}} \simeq \sup \left\{ \int_X fg d\mu \colon |||g|||_{L^{p',1}} \leq 1 \right\},$$

and the right hand side is a norm on $L^{p,\infty}$ that is comparable to $|||\cdot|||_{L^{p,\infty}}$. In this question, we will adapt the approach taken there, and show that if $p \in (1,\infty)$, $r \in [1,\infty)$, then

$$|||f|||_{L^{p,r}} \simeq \sup \left\{ \int_X fg \, d\mu \colon |||g|||_{L^{p',r'}} \leq 1 \right\}.$$

In particular, the right hand side defines a norm on $L^{p,r}$ that is comparable to the quasi-norm $|||\cdot|||_{L^{p,r}}$ on $L^{p,r}$. (With a bit more work one can also show that the dual space of $L^{p,r}$ is $L^{p',r'}$, when $p \in (1,\infty)$, $r \in [1,\infty)$. See also Stein and Weiss' *Introduction to Fourier Analysis*, Chapter V.3, for another approach of defining a norm on $L^{p,r}$, which proceeds via Hardy's inequality.)

(a) Show that if $p \in (0,\infty)$ and $r \in (0,\infty)$, then

$$|||f|||_{L^{p,r}} = \left(\int_0^\infty \left[t^{1/p} f^*(t) \right]^r \frac{dt}{t} \right)^{1/r}$$

where f^* is the decreasing rearrangement of f, namely the unique decreasing non-negative right-continuous function on $[0,\infty)$ so that

$$\mu\{|f| > \alpha\} = |\{|f^*| > \alpha\}|$$

for all $\alpha > 0$. (Hint: Approximate by simple functions.)

(b) Show that

$$\left| \int_X fg d\mu \right| \leq \int_0^\infty f^*(t) g^*(t) dt.$$

Hence show that if $p \in [1,\infty]$ and $r \in [1,\infty]$, then

$$\left| \int_X fg d\mu \right| \leq |||f|||_{L^{p,r}} |||g|||_{L^{p',r'}}.$$

(Hint: For the first part, it suffices to consider non-negative f and g's. Use

$$f(x) = \int_0^\infty \chi_{f(x)\geq u} du$$

and similarly for g, to evaluate the integral on the left hand side; then manipulate using Fubini's theorem, and note that

$$\begin{aligned} & d\mu\{x \colon f(x) \geq u \text{ and } g(x) \geq v\} \\ \leq & \min\{d\mu\{x \colon f(x) \geq u\}, d\mu\{x \colon g(x) \geq v\}\} \\ = & \min\{|\{t \colon f^*(t) \geq u\}|, |\{t \colon g^*(t) \geq v\}|\} \\ = & |\{t \colon f^*(t) \geq u \text{ and } g^*(t) \geq v\}| \end{aligned}$$

the last line following because either the set $\{t \colon f^*(t) \geq u\}$ is contained in $\{t \colon g^*(t) \geq v\}$, or vice versa. Reverse the manipulation done earlier, but to f^* and g^* instead of to f and g, to finish the proof of the first inequality. The second part now follows from the first part by Hölder's inequality.)

(c) Suppose now $p \in (1,\infty)$ and $r \in [1,\infty)$. Let f be a non-negative function in $L^{p,r}$ with $|||f|||_{L^{p,r}} = 1$. For $j \in \mathbb{Z}$, let $E_j = \{x \colon 2^j < f(x) \leq 2^{j+1}\}$. Let also

$$g(x) = \sum_{j\in\mathbb{Z}} \left(2^j \mu(E_j)^{1/p}\right)^{r-1} \mu(E_j)^{-1/p'} \chi_{E_j}(x).$$

Show that $|||g|||_{L^{p',r'}} \lesssim 1$, and that $\int_X fg d\mu \gtrsim 1$. Together with the previous part, we prove the desired comparison of $|||f|||_{L^{p,r}}$ with a genuine norm.
(Hint: Note that

$$g(x) \leq \sum_{k\in\mathbb{Z}} 2^k \chi_{F_k} \quad \text{where} \quad F_k = \bigcup_{2^{k-1} < \left(2^j \mu(E_j)^{1/p}\right)^{r-1} \mu(E_j)^{-1/p'} \leq 2^k} E_j.$$

We want to show that $|||g|||_{L^{p',r'}} \lesssim 1$. Suppose now $r = 1$. It suffices to show $\mu(F_k) \lesssim 2^{-kp'}$ for all $k \in \mathbb{Z}$. But since $|||f|||_{L^{p,r}} = 1$, we have $\mu(E_j) \lesssim 2^{-jp}$ for all $j \in \mathbb{Z}$. Hence for any $k \in \mathbb{Z}$, we have

$$F_k = \bigcup_{\mu(E_j) \simeq 2^{-kp'}} E_j \subset \bigcup_{2^{-jp} \lesssim 2^{-kp'}} E_j,$$

which shows that

$$\mu(F_k) \leq \sum_{2^{-jp} \lesssim 2^{-kp'}} \mu(E_j) \leq \sum_{2^{-jp} \lesssim 2^{-kp'}} 2^{-jp} \lesssim 2^{-kp'},$$

as desired.
Next suppose $r \in (1,\infty)$. If $r = p$ then $g(x) = \sum_{j\in\mathbb{Z}} 2^{jp'} \chi_{E_j} \lesssim f(x)^{p'/p}$, so

$|||g|||_{L^{p',r'}} = \|g\|_{L^{p'}} \lesssim \|f\|_{L^p} = |||f|||_{L^{p,r}} = 1$. Hence we may assume $r \neq p$. We want to show that $\sum_{k\in\mathbb{Z}} 2^{kr'}\mu(F_k)^{r'/p'} \lesssim 1$. But when $r \neq p$,

$$F_k = \bigcup_{\mu(E_j)\simeq(2^k 2^{-j(r-1)})^{\frac{p}{r-p}}} E_j,$$

so

$$\mu(F_k) \leq \sum_{\mu(E_j)\simeq(2^k 2^{-j(r-1)})^{\frac{p}{r-p}}} \mu(E_j).$$

The right hand side is a sum of a subset of an essentially geometric series, so

$$\mu(F_k)^{\frac{r'}{p'}} \lesssim \sum_{\mu(E_j)\simeq(2^k 2^{-j(r-1)})^{\frac{p}{r-p}}} \mu(E_j)^{\frac{r'}{p'}}.$$

(We used $p, r \in (1,\infty)$ here to ensure that the exponent r'/p' is finite; indeed the implicit constant here blows up like $1/(p-1)$ as $p \to 1^+$.) Since $\mu(E_j)^{\frac{r'}{p'}-\frac{r}{p}} = \mu(E_j)^{\frac{p-r}{p}r'} = 2^{-kr'}2^{jr}$ when $\mu(E_j) \simeq (2^k 2^{-j(r-1)})^{\frac{p}{r-p}}$, the right hand side above is bounded by

$$2^{-kr'} \sum_{\mu(E_j)\simeq(2^k 2^{-j(r-1)})^{\frac{p}{r-p}}} 2^{jr}\mu(E_j)^{\frac{r}{p}}.$$

It follows that

$$\sum_{k\in\mathbb{Z}} 2^{kr'}\mu(F_k)^{\frac{r'}{p'}} \lesssim \sum_{j\in\mathbb{Z}} 2^{jr}\mu(E_j)^{\frac{r}{p}} \lesssim 1,$$

as desired.)

7. Show that there is no norm on $L^{1,\infty}$ that is comparable to $|||\cdot|||_{L^{1,\infty}}$. (Hint: We have

$$\left|\left|\left| \sum_{n=1}^{N} \frac{1}{|x-n|} \right|\right|\right|_{L^{1,\infty}} \simeq N\log N, \quad \text{while} \quad \sum_{n=1}^{N} \left|\left|\left| \frac{1}{|x-n|} \right|\right|\right|_{L^{1,\infty}} \simeq N.$$

Obtain a contradiction as $N \to \infty$.)

8. Following the notes for Lecture 8, establish the cases of the Marcinkiewicz interpolation theorem when one of the p_i's is infinite, and/or when one of the q_i's is infinite. (Hint: There are 4 cases to consider:

   (i) $p_0, p_1, q_0 \in (0,\infty)$ and $q_1 = \infty$;
   (ii) $p_0, q_0, q_1 \in (0,\infty)$ and $p_1 = \infty$;
   (iii) $p_0, q_0 \in (0,\infty)$ and $p_1 = q_1 = \infty$;
   (iv) $p_0, q_1 \in (0,\infty)$ and $p_1 = q_0 = \infty$.

   For each of the 4 cases above, one needs to consider two subcases, namely $r \in (0,\infty)$ and $r = \infty$. For case (i), the key is to observe that for $\varepsilon < 1/\beta$, if $2^k W_k^{1/p_1} \gtrsim 2^j c_{k,j}$, then $k\alpha \geq j\beta$; indeed, if $j\beta > k\alpha$, then for $\varepsilon < 1/\beta$, we have

$$\left(2^{\frac{1}{\beta}-\varepsilon}\right)^{k\alpha} \leq \left(2^{\frac{1}{\beta}-\varepsilon}\right)^{j\beta},$$

which rearranges to $(2^k)^{\frac{\alpha}{\beta}} \leq 2^j 2^{-\varepsilon(j\beta - k\alpha)} = 2^j c_{k,j}$, so

$$2^k W_k^{1/p_1} = (2^k W_k^{1/p})^{\frac{p}{p_1}} (2^k)^{1-\frac{p}{p_1}} \lesssim (2^k)^{\frac{\alpha}{\beta}} \leq 2^j c_{k,j},$$

where in the second inequality we used that $2^k W_k^{1/p} \lesssim 1$, and that

$$\frac{\alpha}{\beta} = \frac{p(\frac{1}{p_0} - \frac{1}{p_1})}{q(\frac{1}{q_0} - \frac{1}{\infty})} = \frac{p(\frac{1}{p} - \frac{1}{p_1})}{q(\frac{1}{q} - \frac{1}{\infty})} = 1 - \frac{p}{p_1};$$

in the penultimate equality we used the fact that $(1/p_0, 1/q_0)$, $(1/p_1, 1/q_1)$ and $(1/p, 1/q)$ are collinear. This proves the key observation, which in turn implies that $\mu\{|Tf_k| \geq c_{k,j} 2^j\} = 0$ whenever $k\alpha < j\beta$, so

$$\begin{aligned} \mu\{|Tf| > 2^j\} &\leq \sum_{k\colon k\alpha \geq j\beta} \mu\{|Tf_k| \geq c_{k,j} 2^j\} \\ &\leq \sum_{k\colon k\alpha \geq j\beta} (c_{k,j}^{-1} 2^{-j} 2^k W_k^{1/p_0})^{q_0} \end{aligned}$$

for all $j \in \mathbb{Z}$, and one can finish the proof in this case as in the lecture notes.)

9. (a) Prove the following Phragmén-Lindelöf principle: Suppose f is a holomorphic function on the right half-space $H := \{\operatorname{Re} z > 0\}$ that extends continuously to the closure $\overline{H}$ of H. Assume $|f(z)| \leq 1$ on the boundary of H, and that there exist $\alpha < 1$, and constants C, c, such that

$$|f(z)| \leq C e^{c|z|^\alpha}$$

for all $z \in H$. Then $|f(z)| \leq 1$ on the half-space H.
(Hint: It is crucial that $\alpha < 1$ here, for this allows one to rule out the 'enemy' e^z. We turn this enemy to our advantage: for $\varepsilon > 0$, consider

$$g_\varepsilon(z) := f(z) e^{-\varepsilon z^\beta}$$

where $\beta \in (\alpha, 1)$ and z^β is defined using the principal branch of logarithm. Then $g_\varepsilon(z)$ tends to zero as z tends to infinity within the closed half-plane. Thus the maximum modulus principle shows that $|g_\varepsilon(z)| \leq 1$ for all z in the closed half-plane. It remains to let $\varepsilon \to 0^+$ to obtain the desired conclusion.)

(b) Prove the following extension of the result in the previous part: Suppose f is a holomorphic function on the half-space $H := \{\operatorname{Re} z > 0\}$ that extends continuously to $\overline{H} \setminus \{0\}$. Assume $|f(z)| \leq 1$ on the boundary of H except possibly at 0, and that there exist $\alpha < 1$, and constants C, c, such that

$$|f(z)| \leq C e^{c(|z|^\alpha + |z|^{-\alpha})}$$

for all $z \in H$. Then $|f(z)| \leq 1$ on the half-space H.
(Hint: Consider

$$g_\varepsilon(z) := f(z) e^{-\varepsilon(z^\beta + z^{-\beta})}$$

instead; note that $g_\varepsilon(z) \to 0$ as $z \to 0$ within $\overline{H}$.)

(c) Prove the following three-lines lemma: Suppose F is a holomorphic function on the strip $S := \{0 < \operatorname{Re} z < 1\}$ that extends continuously to the closure of the strip. Assume $|F(z)| \leq A_0$ when $\operatorname{Re} z = 0$, and $|F(z)| \leq A_1$ when $\operatorname{Re} z = 1$. If that there exist $\alpha < 1$, and constants C, c, such that

$$|F(z)| \leq Ce^{ce^{\pi\alpha|z|}}$$

for all $z \in S$, then $|F(z)| \leq A_0^{1-\operatorname{Re} z} A_1^{\operatorname{Re} z}$ on the strip S.
(Hint: By considering $A_0^{-(1-z)} A_1^{-z} F(z)$ in place of $F(z)$, we may assume $A_0 = A_1 = 1$. The conformal map $z \mapsto w$ defined by $w = -ie^{\pi i z}$ maps S conformally onto the right half-space H, with inverse $z = \frac{1}{\pi i}\operatorname{Log}(iw)$. It remains to apply the result of the previous part to the holomorphic function $f(w) = F(\frac{1}{\pi i}\operatorname{Log}(iw))$ defined for $w \in H$. Another way of presenting the same proof is to directly consider $F(z)\exp\left(-\varepsilon \cosh(\pi i(z - \frac{1}{2})\beta)\right)$ for $z \in S$.)

10. Let (X, μ), (Y, ν) be measure spaces. Let $0 < p_0, p_1, q_0, q_1 \leq \infty$, and

$$T\colon (L^{p_0} + L^{p_1})(X) \to (L^{q_0} + L^{q_1})(Y)$$

be a linear operator. Suppose there exist constants A_0, A_1 such that

$$\|Tf\|_{L^{q_0}} \leq A_0 \|f\|_{L^{p_0}} \quad \text{and} \quad \|Tf\|_{L^{q_1}} \leq A_1 \|f\|_{L^{p_1}}$$

for all $f \in L^{p_0} \cap L^{p_1}$. Show that if $\theta \in (0,1)$ and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

then

$$\|Tf\|_{L^q} \leq A_0^{1-\theta} A_1^{\theta} \|f\|_{L^p} \quad \text{for all } f \in L^p.$$

This extends the Riesz-Thorin complex interpolation theorem to Lebesgue exponents below 1. (Hint: Pick $q_- \in (0,\infty)$ such that $q_- < \min\{q_0, q_1\}$. Let $r \in (0,\infty)$ be such that $\frac{1}{r} = \frac{1}{q_-} - \frac{1}{q}$. For all $f \in L^p$, we write

$$\|Tf\|_{L^q} = \| |Tf|^{q_-} \|_{L^{q/q_-}}^{1/q_-} = \left(\sup_{\substack{g \in L^r \text{ simple} \\ \|g\|_{L^r} = 1}} \int |Tf|^{q_-} |g|^{q_-} \right)^{1/q_-}$$

so that

$$A_0^{-(1-\theta)} A_1^{-\theta} \|Tf\|_{L^q} = \left(\sup_{\substack{g \in L^r \text{ simple} \\ \|g\|_{L^r} = 1}} \int |A_0^{-(1-\theta)} A_1^{-\theta} Tf(x) g(x)|^{q_-} dx \right)^{1/q_-}.$$

Now given simple functions f and g, let $\{f_z\}$, $\{g_z\}$ be holomorphic families of functions on $\{0 \leq \operatorname{Re} z \leq 1\}$ such that $f_\theta = f$, $g_\theta = g$,

$$\|f_z\|_{L^{p_0}} \leq \|f\|_{L^p} \quad \text{and} \quad \|g_z\|_{L^{r_0}} \leq \|g\|_{L^r} \quad \text{when } \operatorname{Re} z = 0,$$

$$\|f_z\|_{L^{p_1}} \leq \|f\|_{L^p} \quad \text{and} \quad \|g_z\|_{L^{r_1}} \leq \|g\|_{L^r} \quad \text{when } \operatorname{Re} z = 1,$$

where

$$\frac{1}{r_0} = \frac{1}{q_-} - \frac{1}{q_0}, \quad \frac{1}{r_1} = \frac{1}{q_-} - \frac{1}{q_1}.$$

In other words,

$$f_z(x) = \frac{f(x)}{|f(x)|} \frac{|f(x)|^{p\left(\frac{1-z}{p_0}+\frac{z}{p_1}\right)}}{\|f\|_{L^p}^{p\left(\frac{1-z}{p_0}+\frac{z}{p_1}\right)}} \|f\|_{L^p}, \quad g_z(x) = \frac{g(x)}{|g(x)|} \frac{|g(x)|^{r\left(\frac{1-z}{r_0}+\frac{z}{r_1}\right)}}{\|g\|_{L^r}^{r\left(\frac{1-z}{r_0}+\frac{z}{r_1}\right)}} \|g\|_{L^r}.$$

Then for each $\varepsilon > 0$, the following function of z

$$\int |A_0^{-(1-z)} A_1^{-z} e^{\varepsilon z^2} T f_z(x) g_z(x)|^{q_-} dx$$

is subharmonic in the strip $\{0 < \operatorname{Re} z < 1\}$, continuous up to the boundary of the strip and decays to 0 as z tends to infinity along the strip. Now invoke the maximum modulus principle for subharmonic functions; letting $\varepsilon \to 0^+$ and then taking supremum over all simple functions f and g, we see that $A_0^{-(1-\theta)} A_1^{-\theta} \|Tf\|_{L^q} \leq \|f\|_{L^p}$ for all $f \in L^p$, if $p \neq \infty$; if $p = \infty$ then an easy adaptation of the above argument works since then $p_0 = p_1 = p$.)

11. Let $0 < p_0, p_1, q_0, q_1 \leq \infty$, and S be the strip $\{0 < \operatorname{Re} z < 1\}$. Suppose $\{T_z\}_{z \in \overline{S}}$ is a family of linear operators mapping compactly supported simple functions on $\mathbb{R}^n$ to locally integrable functions on $\mathbb{R}^n$. Assume that for every compactly supported simple function f, the following conditions are satisfied:

    (a) for almost every $x \in \mathbb{R}^n$, the map $z \mapsto T_z f(x)$ is holomorphic on S and continuous up to $\overline{S}$;

    (b) there exists some locally integrable function $\varphi(x)$ on $\mathbb{R}^n$ such that for every $z \in \overline{S}$, we have $|T_z f(x)| \leq \varphi(x)$ for almost every $x \in \mathbb{R}^n$.

    Suppose further that there exist constants A_0, A_1 such that for all compactly supported simple functions f on $\mathbb{R}^n$, we have

    $$\|T_z f\|_{L^{q_j}} \leq A_j \|f\|_{L^{p_j}} \quad \text{whenever } \operatorname{Re} z = j, \text{ for } j = 0, 1.$$

    Show that for any $\theta \in (0,1)$, and any compactly supported simple functions f on $\mathbb{R}^n$, we have

    $$\|T_\theta f\|_{L^q} \leq A_0^{1-\theta} A_1^{\theta} \|f\|_{L^p},$$

    where

    $$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

    In particular, T_θ extends to a continuous linear map from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. (Hint: As before, pick $q_- \in (0, \infty)$ such that $q_- < \min\{q_0, q_1\}$. Let $r \in (0, \infty)$ be such that $\frac{1}{r} = \frac{1}{q_-} - \frac{1}{q}$. For a compactly supported simple function f, we have

    $$A_0^{-(1-\theta)} A_1^{-\theta} \|T_\theta f\|_{L^q} = \left(\sup_{\substack{g \text{ simple, compactly supported} \\ \|g\|_{L^r}=1}} \int |A_0^{-(1-\theta)} A_1^{-\theta} T_\theta f(x) g(x)|^{q_-} dx \right)^{1/q_-}.$$

Now given compactly supported simple functions f and g, let $\{f_z\}$, $\{g_z\}$ be holomorphic families of functions on $\{0 \leq \operatorname{Re} z \leq 1\}$ such that $f_\theta = f$, $g_\theta = g$,

$$\|f_z\|_{L^{p_0}} \leq \|f\|_{L^p} \quad \text{and} \quad \|g_z\|_{L^{r_0}} \leq \|g\|_{L^r} \quad \text{when } \operatorname{Re} z = 0,$$

$$\|f_z\|_{L^{p_1}} \leq \|f\|_{L^p} \quad \text{and} \quad \|g_z\|_{L^{r_1}} \leq \|g\|_{L^r} \quad \text{when } \operatorname{Re} z = 1,$$

where

$$\frac{1}{r_0} = \frac{1}{q_-} - \frac{1}{q_0}, \quad \frac{1}{r_1} = \frac{1}{q_-} - \frac{1}{q_1}.$$

Then for each $\varepsilon > 0$, the following function of z

$$\int \left| A_0^{-(1-z)} A_1^{-z} e^{\varepsilon z^2} T_z f_z(x) g_z(x) \right|^{q_-} dx$$

is subharmonic in the strip $\{0 < \operatorname{Re} z < 1\}$, continuous up to the boundary of the strip, and decays to 0 as z tends to infinity along the strip. Now apply the maximum modulus principle for subharmonic functions; letting $\varepsilon \to 0^+$ and then taking supremum over all compactly supported simple functions g, we see that $A_0^{-(1-\theta)} A_1^{-\theta} \|T_\theta f\|_{L^q} \leq \|f\|_{L^p}$, as desired.)

12. Let (X, dx), (Y, dy) be measure spaces, and S be the strip $\{0 < \operatorname{Re} z < 1\}$. Let

$$f(x,y) = \sum_{j,k} c_{j,k} \chi_{A_j}(x) \chi_{B_k}(y)$$

be a simple function on $X \times Y$. Suppose $0 < p_0, p_1, q_0, q_1 \leq \infty$, $\theta \in (0,1)$ and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Show that there exists a family of simple functions $\{f_z(x,y)\}_{z \in \overline{S}}$, holomorphic for $z \in S$, continuous and bounded as z varies on $\overline{S}$, such that $f_\theta(x,y) = f(x,y)$, and

$$\|f_z\|_{L^{p_0}_x L^{q_0}_y} \leq \|f\|_{L^p_x L^q_y} \quad \text{when } \operatorname{Re} z = 0,$$

$$\|f_z\|_{L^{p_1}_x L^{q_1}_y} \leq \|f\|_{L^p_x L^q_y} \quad \text{when } \operatorname{Re} z = 1.$$

Here

$$\|f\|_{L^p_x L^q_y} := \| \|f(x,y)\|_{L^q(dy)} \|_{L^p(dx)}.$$

This allows one to formulate a vector-valued complex interpolation theorem involving the anisotropic Lebesgue spaces $L^p_x L^q_y$. (Hint: Just take

$$f_z(x,y) = \frac{f(x,y)}{|f(x,y)|} \frac{|f(x,y)|^{q\left(\frac{1-z}{q_0} + \frac{z}{q_1}\right)}}{\|f(x,y)\|_{L^q_y}^{q\left(\frac{1-z}{q_0} + \frac{z}{q_1}\right)}} \frac{\|f(x,y)\|_{L^q_y}^{p\left(\frac{1-z}{p_0} + \frac{z}{p_1}\right)}}{\|f(x,y)\|_{L^p_x L^q_y}^{p\left(\frac{1-z}{p_0} + \frac{z}{p_1}\right)}} \|f(x,y)\|_{L^p_x L^q_y};$$

c.f. Question 10.)

13. The goal of this question is to prove a bilinear complex interpolation theorem. Let (X_1, μ_1), (X_2, μ_2), (Y, ν) be measure spaces. Let $0 < p_0, p_1, q_0, q_1, r_0, r_1 \leq \infty$, and

$$T\colon (L^{p_0} + L^{p_1})(X_1) \times (L^{q_0} + L^{q_1})(X_2) \to (L^{r_0} + L^{r_1})(Y)$$

be a bilinear operator. Suppose there exist constants A_0, A_1 such that

$$\|T(f_1, f_2)\|_{L^{r_0}} \leq A_0 \|f_1\|_{L^{p_0}} \|f_2\|_{L^{q_0}} \quad \text{and} \quad \|T(f_1, f_2)\|_{L^{r_1}} \leq A_1 \|f_1\|_{L^{p_1}} \|f_2\|_{L^{q_1}}$$

for all $f_1 \in L^{p_0} \cap L^{p_1}(X_1)$ and $f_2 \in L^{q_0} \cap L^{q_1}(X_2)$. Show that if $\theta \in (0, 1)$ and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1},$$

then

$$\|T(f_1, f_2)\|_{L^r} \leq A_0^{1-\theta} A_1^{\theta} \|f_1\|_{L^p} \|f_2\|_{L^q} \quad \text{for all } f_1 \in L^p,\ f_2 \in L^q.$$

Also generalize this to a multilinear complex interpolation theorem. (Hint: Mimic Question 10.)

14. In the last question, we saw an easy extension of the complex interpolation theorem to the multilinear setting. In this question, we will see that the real method of interpolation does not extend to the multilinear setting the way one might naively expect. In particular, we show that there exists a bilinear operator T, such that T is continuous from (say) $L^2 \times L^2$ into $L^{2,\infty}$, continuous from $L^\infty \times L^1$ into L^∞, but does not map $L^p \times L^{p'}$ continuously into L^p, even if $p' \leq p$. So two endpoints are not enough, if we only begin with weak-type hypothesis and want a strong-type conclusion! (The following example is due to Strichartz.)
We will consider the measure space $X = (0, \infty)$ with Lebesgue measure dx. The bilinear operator involved is given by

$$T(f, g)(x) = \int_0^\infty f(xy) g(y) dy.$$

(a) Show that

$$\|T(f, g)\|_{L^{p,\infty}} \leq \|f\|_{L^p} \|g\|_{L^{p'}} \quad \text{for all } 1 \leq p \leq \infty.$$

Indeed, $|T(f, g)(x)| \leq x^{-1/p} \|f\|_{L^p} \|g\|_{L^{p'}}$. In particular, T is continuous from (say) $L^2 \times L^2$ into $L^{2,\infty}$, continuous from $L^\infty \times L^1$ into L^∞.

(b) Let $p \in [1, \infty)$. Show that for any non-negative measurable function g on $(0, \infty)$, we have

$$\sup_{\|f\|_{L^p} \leq 1} \|T(f, g)\|_{L^p} = \int_0^\infty y^{-1/p} g(y) dy.$$

This shows that

$$\sup_{\|f\|_{L^p} \leq 1,\, \|g\|_{L^{p'}} \leq 1} \|T(f, g)\|_{L^p} = \infty.$$

Hence T is not continuous from $L^p \times L^{p'}$ to L^p. (Hint: By scaling, we may assume that $\int_0^\infty y^{-1/p} g(y) dy = 1$. The triangle inequality shows that

$$\sup_{\|f\|_{L^p} \leq 1} \|T(f,g)\|_{L^p} \leq 1.$$

On the other hand, let $\alpha \in (0,1)$ be sufficiently close to 1, and let $\varepsilon \in (0,1)$ be sufficiently close to 0. Let $f_\varepsilon(x) = x^{-1/p} \chi_{[\varepsilon, \varepsilon^{-1}]}(x)$. Then

$$\|f_\varepsilon\|_{L^p} = [2 \log(\varepsilon^{-1})]^{1/p},$$

and

$$T(f_\varepsilon, g)(x) = x^{-1/p} \int_{\varepsilon x^{-1}}^{\varepsilon^{-1} x^{-1}} y^{-1/p} g(y) dy.$$

For $x \in [\varepsilon^\alpha, \varepsilon^{-\alpha}]$, we have

$$T(f_\varepsilon, g)(x) \geq x^{-1/p} \int_{\varepsilon^{1-\alpha}}^{\varepsilon^{\alpha-1}} y^{-1/p} g(y) dy.$$

Thus

$$\|T(f_\varepsilon, g)\|_{L^p} \geq \|x^{-1/p}\|_{L^p[\varepsilon^\alpha, \varepsilon^{-\alpha}]} \int_{\varepsilon^{1-\alpha}}^{\varepsilon^{\alpha-1}} y^{-1/p} g(y) dy,$$

which gives

$$\|T(f_\varepsilon, g)\|_{L^p} \geq \alpha^{1/p} \|f_\varepsilon\|_{L^p} \int_{\varepsilon^{1-\alpha}}^{\varepsilon^{\alpha-1}} y^{-1/p} g(y) dy.$$

By taking $\alpha \in (0,1)$ sufficiently close to 1, and $\varepsilon \in (0,1)$ sufficiently close to 0, we can make the right hand side $> (1-\delta)\|f_\varepsilon\|_{L^p}$ for any $\delta > 0$. Thus

$$\sup_{\|f\|_{L^p} \leq 1} \|T(f,g)\|_{L^p} \geq 1$$

as well, as desired.)

15. Suppose $0 < p < \infty$, and $h \in L^{p_0}(\mathbb{R}^n)$ for some $0 < p_0 \leq p$. Show that if $M^\sharp h \in L^p(\mathbb{R}^n)$, then $Mh \in L^p(\mathbb{R}^n)$, and

$$\|Mh\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|M^\sharp h\|_{L^p(\mathbb{R}^n)}.$$

Here Mh is the Hardy-Littlewood maximal function of h. In particular, we have $h \in L^p(\mathbb{R}^n)$, and $\|h\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|M^\sharp h\|_{L^p(\mathbb{R}^n)}$.
(Hint: The key is a relative distribution inequality. For any $n \geq 1$, we claim that there exists $b_n \in (0,1)$, such that for any $b, c > 0$ with $b \leq b_n$, we have

$$|\{x \in \mathbb{R}^n \colon Mh(x) > \alpha, M^\sharp h(x) \leq c\alpha\}| \lesssim_n c |\{x \in \mathbb{R}^n \colon Mh(x) > b\alpha\}|$$

If this is true, then by taking c sufficiently small, we can use Question 18 of Homework 3 to conclude the proof.

To prove the above relative distributional inequality, let $b \in (0,1)$ first. Let $h \in L^p(\mathbb{R}^n)$. Decompose the open set $\{x \in \mathbb{R}^n \colon Mh(x) > b\alpha\}$ into an essentially disjoint union of Whitney cubes $\{Q\}$, so that the distance of each Q from the complement of this set is bounded by 4 times the diameter of Q. Now since $\{x \in \mathbb{R}^n \colon Mh(x) > \alpha, M^\sharp h(x) \leq c\alpha\}$ is a subset of $\{x \in \mathbb{R}^n \colon Mh(x) > b\alpha\}$, we just need to show that for each Whitney cube Q as above, we have

$$|\{x \in Q \colon Mh(x) > \alpha, M^\sharp h(x) \leq c\alpha\}| \lesssim_n c|Q|.$$

This inequality would be trivial if the set on the left hand side were empty. So let's assume there exists a point $x_0 \in Q$ such that $M^\sharp h(x_0) \leq c\alpha$. Now let $\tilde{Q}$ be any cube that intersects Q and that has diameter at least that of Q. Then $20\tilde{Q}$ will contain a point y where $Mh(y) \leq b\alpha$. Hence $⨍_{\tilde{Q}} |h| \leq 20^n b\alpha$ for all such cubes $\tilde{Q}$. If $x \in Q$ and $Mh(x) > \alpha$, then by taking $b < 20^{-n}$, we see that $M(h\chi_{3Q})(x) > \alpha$. We also have $⨍_{3Q} |h| \leq 20^n b\alpha$. Thus

$$\{x \in Q \colon Mh(x) > \alpha, M^\sharp h(x) \leq c\alpha\} \subset \left\{x \in Q \colon M\left(h\chi_{3Q} - ⨍_{3Q} h\right)(x) > (1-20^n b)\alpha\right\},$$

whose measure is bounded by

$$\frac{C_n}{(1-20^n b)\alpha} \int_Q |h\chi_{3Q}(y) - h_{3Q}| dy \leq \frac{C_n}{(1-20^n b)\alpha} |3Q| M^\sharp h(x_0) \leq \frac{3^n C_n}{(1-20^n b)} c|Q|$$

where C_n is the constant arising in the weak-type (1,1) bound of $M \colon L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$. This proves the desired relative distributional inequality.)

16. The goal of this question is to prove a complex interpolation theorem involving the Hardy space $\mathcal{H}^1$ on $\mathbb{R}^n$ (see Homework 7 for its definition). Let $p_1 \in [1,\infty]$. Suppose $\{T_z\}_{z \in \overline{S}}$ is a family of continuous linear operators from $L^{p_1}(\mathbb{R}^n)$ to $L^1_{\text{loc}}(\mathbb{R}^n)$, analytic in the sense that for every f and g that are bounded measurable with compact support, the map $z \mapsto \int_{\mathbb{R}^n} T_z f \cdot g dx$ is holomorphic on S, continuous up to $\overline{S}$ and bounded on $\overline{S}$. Let $q_0, q_1 \in [1,\infty]$. Assume

$$\|T_z f\|_{L^{q_0}} \leq A_0 \|f\|_{\mathcal{H}^1} \quad \text{whenever } \operatorname{Re} z = 0 \text{ and } f \in \mathcal{H}^1 \cap L^{p_1},$$

$$\|T_z f\|_{L^{q_1}} \leq A_1 \|f\|_{L^{p_1}} \quad \text{whenever } \operatorname{Re} z = 1 \text{ and } f \in L^{p_1}.$$

Then for any $\theta \in (0,1)$, we have

$$\|T_\theta f\|_{L^q} \lesssim A_0^{1-\theta} A_1^\theta \|f\|_{L^p} \quad \text{for all } f \in L^p \cap L^{p_1},$$

where

$$\frac{1}{p} = (1-\theta) + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

In particular, T_θ extends to a bounded linear map from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, with norm $\lesssim A_0^{1-\theta} A_1^\theta$. (Hint: For each $z \in \overline{S}$ and every bounded measurable g with compact support, let $T_z^* g$ be the $L^{p_1'}$ function on $\mathbb{R}^n$, such that

$$\int_{\mathbb{R}^n} f \cdot T_z^* g dx = \int_{\mathbb{R}^n} T_z f \cdot g dx$$

for all $f \in L^{p_1}$. Now let $\theta \in (0,1)$, and let g be a compactly supported simple function on $\mathbb{R}^n$. Let $\{g_z\}$ be a holomorphic family with $g_\theta = g$ and

$$\|g_z\|_{L^{q_0{}'}} \leq \|g\|_{L^{q'}} \quad \text{when } \operatorname{Re} z = 0,$$

$$\|g_z\|_{L^{q_1{}'}} \leq \|g\|_{L^{q'}} \quad \text{when } \operatorname{Re} z = 1.$$

Then $T_z^* g_z$ is an analytic family of L^1_{loc} function, and satisfies the hypothesis of the proposition we used in the lecture notes to prove the complex interpolation theorem for BMO; note in particular that $T_z^* g_z \in L^{p_1{}'}$ for all $z \in \overline{S}$,

$$\|T_z^* g_z\|_{BMO} \leq A_0 \|g\|_{L^{q'}} \quad \text{when } \operatorname{Re} z = 0,$$

$$\|T_z^* g_z\|_{L^{p_1{}'}} \leq A_1 \|g\|_{L^{q'}} \quad \text{when } \operatorname{Re} z = 1.$$

Thus $T_\theta^* g \in L^{p'}$, with

$$\|T_\theta^* g\|_{L^{p'}} \lesssim A_0^{1-\theta} A_1^\theta \|g\|_{L^{q'}}.$$

As a result, if $f \in L^p \cap L^{p_1}$, then

$$\left| \int_{\mathbb{R}^n} T_\theta f \cdot g dx \right| \lesssim A_0^{1-\theta} A_1^\theta \|f\|_{L^p} \|g\|_{L^{q'}}.$$

By density of compactly supported simple functions in $L^{q'}$, this shows that

$$\|T_\theta f\|_{L^{p'}} \lesssim A_0^{1-\theta} A_1^\theta \|f\|_{L^p}$$

for all $f \in L^p \cap L^{p_1}$, as desired.)