Topics in Harmonic Analysis
Lecture 1: The Fourier transform

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Outline

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Fourier series on $\mathbb{T}$: L^2 theory

- Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the unit circle.
- $\{$Functions on $\mathbb{T}\} = \{$periodic functions on $\mathbb{R}\}$.
- The n-th Fourier coefficient of an L^1 function on $\mathbb{T}$ is given by

$$\widehat{f}(n) := \int_{\mathbb{T}} f(x)e^{-2\pi inx}dx, \quad n \in \mathbb{Z}.$$

- A remarkable insight of J. Fourier was that perhaps 'every' function f on $\mathbb{T}$ can be represented by a Fourier series:

$$\sum_{n\in\mathbb{Z}} a_n e^{2\pi inx}.$$

- One can make this rigorous, for instance, when one restricts attention to L^2 functions on $\mathbb{T}$ (which form a Hilbert space).

- The claim is that for every $f \in L^2(\mathbb{T})$, the Fourier series of f converges to f in $L^2(\mathbb{T})$ norm.
- In other words, for every $f \in L^2(\mathbb{T})$, we have

$$\sum_{n=-N}^{N} \widehat{f}(n) e^{2\pi i n x} \to f(x) \qquad \text{in } L^2(\mathbb{T}) \text{ as } N \to \infty.$$

- Underlying this claim are two important concepts, namely *orthogonality* and *completeness*.

- Clearly $\{e^{2\pi inx}\}_{n\in\mathbb{Z}}$ is an orthonormal system on $L^2(\mathbb{T})$, and

$$\widehat{f}(n) = \langle f(x), e^{2\pi inx}\rangle_{L^2(\mathbb{T})}.$$

- General principles about orthogonality then shows that for every $N \in \mathbb{N}$,

$$S_N f(x) := \sum_{|n|\leq N} \widehat{f}(n) e^{2\pi inx}$$

  is the orthogonal projection of f onto the linear subspace V_N of $L^2(\mathbb{T})$ spanned by $\{e^{2\pi inx} : |n| \leq N\}$. In other words,

$$f = (f - S_N f) + S_N f$$

  with $S_N f \in V_N$ and $(f - S_N f) \perp V_N$, and hence

$$\sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^2 \leq \|f\|^2_{L^2(\mathbb{T})}.$$

  (The last inequality is sometimes called Bessel's inequality.)

- The aforementioned property about orthogonal projection can be rephrased as

$$\|f - S_N f\|_{L^2(\mathbb{T})} = \min\left\{\|f - p_N\|_{L^2(\mathbb{T})} : p_N \in \operatorname{span}\{e^{2\pi inx}\}_{|n|\leq N}\right\}.$$

- On the other hand, one can show that the set of all finite linear combinations of $\{e^{2\pi inx}\}_{n\in\mathbb{Z}}$ (i.e. the set of all trigonometric polynomials) is *dense* in $L^2(\mathbb{T})$ (more to follow below).
- Thus $\{e^{2\pi inx}\}_{n\in\mathbb{Z}}$ form a *complete orthonormal system* on $L^2(\mathbb{T})$.
- It follows that for every $f \in L^2(\mathbb{T})$, we have

$$S_N f \to f \qquad \text{in } L^2(\mathbb{T}) \text{ as } N \to \infty.$$

- To see this desired density, we study convolutions and multiplier operators.

Convolutions

- The convolutions of two L^1 functions on $\mathbb{T}$ is another L^1 function, given by

$$f * g(x) = \int_{\mathbb{T}} f(x-y)g(y)dy.$$

- Convolutions are associative and commutative:

$$f * (g * h) = (f * g) * h, \qquad f * g = g * f.$$

- If $f \in L^p(\mathbb{T})$ for some $p \in [1, \infty]$ and $g \in L^1(\mathbb{T})$, then $f * g \in L^p(\mathbb{T})$.
- More generally, if $p, q, r \in [1, \infty]$ and

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$$

then

$$\|f * g\|_{L^r(\mathbb{T})} \leq \|f\|_{L^p(\mathbb{T})} \|g\|_{L^q(\mathbb{T})}.$$

(Young's convolution inequality)

- For $f \in L^1(\mathbb{T})$, let

$$\widehat{f}(n) = \int_{\mathbb{T}} f(x) e^{-2\pi i n x} dx$$

  as before. Then

$$\widehat{f * g}(n) = \widehat{f}(n)\widehat{g}(n)$$

  whenever $f, g \in L^1(\mathbb{T})$.
- So if $K \in L^1(\mathbb{T})$, then the convolution operator

$$f \mapsto f * K$$

  can be understood via the Fourier transform of K: indeed

$$f * K(x) = \sum_{n \in \mathbb{Z}} \widehat{f}(n)\widehat{K}(n) e^{2\pi i n x}$$

  whenever f is in (say) $L^2(\mathbb{T})$.

The Dirichlet and Fejer kernels

- For each $N \in \mathbb{N}$, let D_N and F_N be the Dirichlet and Fejer kernels respectively, defined by

$$\widehat{D_N}(n) = \begin{cases} 1 & \text{if } |n| \leq N \\ 0 & \text{if } |n| > N \end{cases}, \qquad \widehat{F_N}(n) = \begin{cases} 1 - \frac{|n|}{N} & \text{if } |n| \leq N \\ 0 & \text{if } |n| > N \end{cases}.$$

Then

$$D_N(x) = \sum_{|n| \leq N} e^{2\pi i n x}$$

$$F_N(x) = \sum_{|n| \leq N} \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x}$$

- If $f \in L^1(\mathbb{T})$, then
$$S_N f = f * D_N,$$
and
$$\frac{S_0 f + S_1 f + \cdots + S_{N-1} f}{N} = f * F_N.$$
- One has a closed formula for F_N: indeed
$$F_N(x) = \frac{1}{N}\left(\frac{\sin N\pi x}{\sin \pi x}\right)^2.$$
- From this we see that $F_N \geq 0$ for all N,
$$\|F_N\|_{L^1} = \widehat{F_N}(0) = 1,$$
and
$$\int_{\delta \leq |y| \leq 1/2} F_N(y) dy \to 0 \quad \text{as } N \to \infty.$$
Thus $\{F_N\}_{N \in \mathbb{N}}$ form a family of *good kernels*.

- As a result, if f is continuous on $\mathbb{T}$, then $f * F_N \to f$ uniformly on $\mathbb{T}$.
- Hence by approximating by continuous functions, we see that trigonometric polynomials are dense in $L^2(\mathbb{T})$.
- This establishes in full the elementary L^2 theory of the Fourier transform on $\mathbb{T}$. In particular, now we have Plancherel's theorem:

$$\sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^2 = \|f\|_{L^2(\mathbb{T})}^2$$

whenever $f \in L^2(\mathbb{T})$.

Pointwise convergence of Fourier series

- We can now prove the Riemann-Lebesgue lemma: for $f \in L^1(\mathbb{T})$, we have

$$\sup_{n \in \mathbb{Z}} |\widehat{f}(n)| \leq \|f\|_{L^1(\mathbb{T})}.$$

Indeed $\widehat{f}(n) \to 0$ as $n \to \pm\infty$.

- This allows one to show that

$$\lim_{N \to \infty} \sum_{|n| \leq N} \widehat{f}(n) e^{2\pi i n x} = f(x)$$

whenever f is Hölder continuous of some positive order at $x \in \mathbb{T}$ (actually we only need a Dini condition at x: indeed the condition

$$\int_{|t| \leq 1/2} \frac{|f(x+t) - f(x)|}{|t|} dt < \infty$$

will suffice.)

- On the other hand, there exists a continuous function on $\mathbb{T}$ whose Fourier series diverges at say $0 \in \mathbb{T}$. So mere continuity of f does not guarantee everywhere pointwise convergence of Fourier series!
- On the pathological side, there even exists an L^1 function on $\mathbb{T}$, whose Fourier series diverges everywhere (Kolmogorov).
- However, a remarkable theorem of Carleson says that the Fourier series of an L^2 function on $\mathbb{T}$ converges pointwise almost everywhere (a.e.).
  (Later Hunt showed the same for $L^p(\mathbb{T})$, $1 < p < \infty$.)
- We will come back to this briefly towards the end of this lecture.

Other modes of convergence

- To sum up, we have considered L^2 norm convergence, pointwise convergence and a.e. pointwise convergence of Fourier series.
- Other modes of convergence of Fourier series also give rise to interesting questions.
- Examples include L^p norm convergence, uniform convergence, absolute convergence, and various summability methods.
- If $f \in C^2(\mathbb{T})$, then clearly the Fourier series of f converges absolutely and uniformly (since $|\widehat{f}(n)| \lesssim |n|^{-2}$).
- However, indeed the Fourier series of f converges absolutely and uniformly already, as long as f is Hölder continuous of some order $> 1/2$ on $\mathbb{T}$.
- The question of L^p norm convergence on $\mathbb{T}$, for $1 < p < \infty$, is also well understood, and will be considered in Lecture 4.
- It is interesting to note that an analogous question in higher dimensions (for $\mathbb{T}^n$, $n > 1$) is much deeper, and is related to some excellent open problems in the area.

Multiplier operators: a prelude

- Given a bounded function $m\colon \mathbb{Z} \to \mathbb{C}$, the map

$$f(x) \mapsto T_m f(x) := \sum_{n\in\mathbb{Z}} m(n)\widehat{f}(n)e^{2\pi i n x}$$

  defines a bounded linear operator on $L^2(\mathbb{T})$.
- Examples: convolution with the Dirichlet or the Fejer kernels.
- The analysis of such operators often benefit by taking the inverse Fourier transform of m, as we have seen in the case of convolution with the Fejer kernel.
- We will be interested, for instance, in the boundedness of many multiplier operators on $L^p(\mathbb{T})$ for various $p \in [1, \infty]$.
- e.g. The uniform L^p boundedness of convolutions with the Dirichlet kernels is crucial in the discussion of L^p norm convergence of the Fourier series of a function in $L^p(\mathbb{T})$.
- We note in passing that all multiplier operators commute with translations (and so do all convolution operators).

The role played by the group of translations

- Let's take a step back and look at Fourier series on $L^2(\mathbb{T})$.
- There we expand functions in terms of complex exponentials $\{e^{2\pi inx}\}_{n\in\mathbb{Z}}$.
- But why complex exponentials?
- An explanation can be given in terms of the underlying group structure on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.
- Recall $\mathbb{T}$ is an abelian group under addition.
- Thus if $y \in \mathbb{T}$, then the operator $\tau_y \colon L^2(\mathbb{T}) \to L^2(\mathbb{T})$, defined by

  $$\tau_y f(x) = f(x+y),$$

  is a unitary operator on $L^2(\mathbb{T})$, and the group homomorphism

  $$y \in \mathbb{T} \mapsto \tau_y \in B(L^2(\mathbb{T}))$$

  is continuous if we endow the strong topology on $B(L^2(\mathbb{T}))$.

- Thus the map $y \mapsto \tau_y$ defines a unitary representation of the compact group $\mathbb{T}$ on the (complex) Hilbert space $L^2(\mathbb{T})$.
- The Peter-Weyl theorem then provides a splitting of $L^2(\mathbb{T})$ into an orthogonal direct sum of irreducible finite-dimensional representations.
- Since $\mathbb{T}$ is abelian, all irreducible finite-dimensional representations are 1-dimensional, by Schur's lemma.
- In other words, there exists an orthonormal family of functions $\{f_n\}_{n\in\mathbb{Z}}$ of $L^2(\mathbb{T})$, such that

$$L^2(\mathbb{T}) = \bigoplus_{n\in\mathbb{Z}} \mathbb{C} f_n,$$

  and such that f_n is an eigenfunction of τ_y for all $y \in \mathbb{T}$.

- For $n \in \mathbb{Z}$, let $\chi_n(y)$ be the eigenvalue of τ_y on f_n, i.e. let $\chi_n \colon \mathbb{T} \to \mathbb{C}^\times$ be a continuous function such that

$$\tau_y f_n = \chi_n(y) f_n \quad \text{for all } y \in \mathbb{T}.$$

- Then $\chi_n(y + y') = \chi_n(y)\chi_n(y')$ for all $y, y' \in \mathbb{T}$, i.e. χ_n is a character on $\mathbb{T}$.
- But if $\chi \colon \mathbb{R} \to \mathbb{C}^\times$ is a continuous function with

$$\chi(x + x') = \chi(x)\chi(x')$$

for all $x, x' \in \mathbb{R}$, then there exists $a \in \mathbb{C}$ such that $\chi(x) = e^{ax}$ for all $x \in \mathbb{R}$ (just show that $\chi(x) = e^{ax}$ holds for $x = m x_0 / 2^n$ whenever $m \in \mathbb{Z}$, $n \in \mathbb{N}$ and x_0 is sufficiently close to 0, and use continuity); if in addition χ is periodic with period 1, then the a above must be in $2\pi i \mathbb{Z}$.
- Thus without loss of generality, we may label the f_n's, so that $\chi_n(y) = e^{2\pi i n y}$ for all $n \in \mathbb{Z}$ and all $y \in \mathbb{T}$.

- This gives
$$f_n(x+y) = e^{2\pi i n y} f_n(x)$$
for all $n \in \mathbb{Z}$ and all $x, y \in \mathbb{T}$, so
$$f_n'(x) = 2\pi i n f_n(x),$$
from which we conclude that $f_n(x) = ce^{2\pi i n x}$ for some $|c| = 1$.
- In other words, for each $n \in \mathbb{Z}$, the function $e^{2\pi i n x}$ is an eigenvector of τ_y for all $y \in \mathbb{T}$, with eigenvalue $e^{2\pi i n y}$.
- To recap: we have a family $\{\tau_y\}_{y \in \mathbb{T}}$ of commuting unitary operators on $L^2(\mathbb{T})$, and the complex exponentials $\{e^{2\pi i n x}\}_{n \in \mathbb{Z}}$ provide a simultaneous diagonalization of these operators:
$$\tau_y e^{2\pi i n x} = e^{2\pi i n y} e^{2\pi i n x}$$
for all $n \in \mathbb{Z}$, and all $y \in \mathbb{T}$.
- Note that the orthogonality of the complex exponentials follows from the Peter-Weyl theorem!

- Also, the equation

$$\tau_y e^{2\pi inx} = e^{2\pi iny} e^{2\pi inx}$$

implies that $e^{2\pi inx}$ are eigenfunctions of the derivative operator:

$$\frac{d}{dx} e^{2\pi inx} = 2\pi i n e^{2\pi inx}.$$

This is not too surprising since derivatives are infinitesimal translations!
- Another way of saying the same thing is that

$$\widehat{\tau_y f}(n) = e^{2\pi iny} \widehat{f}(n)$$

for all $y \in \mathbb{T}$ and $n \in \mathbb{Z}$; hence differentiation and multiplication are interwined by the Fourier transform:

$$\widehat{f'}(n) = 2\pi in\widehat{f}(n) \quad \text{if } f \in C^1(\mathbb{T}).$$

- The fact that the Fourier series constitutes a spectral decomposition of the derivative operator is what makes it so powerful in the study of differential equations.

Analogue on $\mathbb{R}^n$

- The Fourier transform of an L^1 function on $\mathbb{R}^n$ is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi} dx, \quad \xi \in \mathbb{R}^n.$$

- We have

$$\|\widehat{f}\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)};$$

  indeed the Fourier transform of an L^1 function is continuous on $\mathbb{R}^n$.
- If $f, g \in L^1(\mathbb{R}^n)$, their convolution is defined by

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y) g(y) dy.$$

- We have $f * g = g * f$ and

$$\widehat{f * g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$$

  if $f, g \in L^1(\mathbb{R}^n)$.

- The space of Schwartz functions on $\mathbb{R}^n$ is defined as the space of all smooth functions, whose derivatives of all orders are rapidly decreasing at infinity. It is denoted $\mathcal{S}(\mathbb{R}^n)$.
- One can restrict the Fourier transform on $\mathcal{S}(\mathbb{R}^n)$; indeed the Fourier transform maps $\mathcal{S}(\mathbb{R}^n)$ into itself.
- If f, g are Schwartz functions on $\mathbb{R}^n$, then Fubini's theorem gives

  $$\int_{\mathbb{R}^n} f(y)\widehat{g}(y)dy = \int_{\mathbb{R}^n} \widehat{f}(\xi)g(\xi)d\xi.$$

  Replacing $g(\xi)$ by $\overline{g}(\xi)e^{2\pi i x\cdot\xi}$, it follows that

  $$f * \overline{\widehat{g}}(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi)\overline{g}(\xi)e^{2\pi i x\cdot\xi}d\xi$$

  for all $x \in \mathbb{R}^n$.

$$f * \overline{\widehat{g}}(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi)\overline{g}(\xi)e^{2\pi i x\cdot\xi}\, d\xi$$

- Applying this with $g(\xi) = e^{-\pi t|\xi|^2}$ (the heat kernel) and letting $t \to 0$, one can show that the Fourier inversion formula holds for Schwartz functions:

  $$f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi)e^{2\pi i x\cdot\xi}\, d\xi$$

  for every $x \in \mathbb{R}^n$, whenever $f \in \mathcal{S}(\mathbb{R}^n)$.
- Hence the Fourier transform defines a *bijection* on Schwartz functions on $\mathbb{R}^n$, and we have

  $$f * g(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi)\widehat{g}(\xi)e^{2\pi i x\cdot\xi}\, d\xi$$

  whenever $f, g \in \mathcal{S}(\mathbb{R}^n)$.

- We now also have

$$\int_{\mathbb{R}^n} f(x)\overline{g(x)}dx = \int_{\mathbb{R}^n} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}d\xi$$

  whenever $f, g \in \mathcal{S}(\mathbb{R}^n)$.
- This allows one to show that the Fourier transform, initially defined on $\mathcal{S}(\mathbb{R}^n)$, extends as a unitary operator on $L^2(\mathbb{R}^n)$, and the Plancherel formula holds:

$$\|\widehat{f}\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$$

  whenever $f \in L^2(\mathbb{R}^n)$.
- A multiplier operator on $\mathbb{R}^n$ is of the form $f \mapsto (m\widehat{f})^{\vee}$ where m is a bounded measurable function on $\mathbb{R}^n$.
- We will come across examples of such in Lecture 2.
- These are automatically bounded on $L^2(\mathbb{R}^n)$. We will study their mapping properties on $L^p(\mathbb{R}^n)$ in Lecture 4.

The groups of modulations and dilations

- $\mathbb{R}^n$ is an abelian group under addition.
- It acts on (say, L^2) functions on $\mathbb{R}^n$ by translation (as in the case of the unit circle $\mathbb{T}$):

$$\tau_y f(x) := f(x+y), \quad y \in \mathbb{R}^n$$

- But it also acts on functions on $\mathbb{R}^n$ by modulation:

$$\Lambda_\xi f(x) := e^{2\pi i x\cdot\xi} f(x), \quad \xi \in \mathbb{R}^n.$$

- The actions are interwined by the Fourier transform $\mathcal{F}$:

$$\mathcal{F}\tau_y = \Lambda_y \mathcal{F} \quad \text{for all } y \in \mathbb{R}^n.$$

- In particular, at least for Schwartz functions f on $\mathbb{R}^n$, we have

$$\widehat{\partial_j f}(\xi) = 2\pi i \xi_j \widehat{f}(\xi), \quad \text{for } 1 \le j \le n.$$

- The multiplicative group $\mathbb{R}^+ = (0, \infty)$ also acts on functions on $\mathbb{R}^n$ by dilations:

$$D_t f(x) := f(tx), \quad t \in \mathbb{R}^+.$$

- It interacts with the Fourier transform as follows:

$$\mathcal{F} D_t = t^{-n} D_{1/t} \mathcal{F} \quad \text{for all } t > 0.$$

- In harmonic analysis we often study operators that commutes with translations.
- Examples include derivative operators (such as $f \mapsto \Delta f$), and convolution operators (such as $f \mapsto f * |x|^{-(n-2)}$).
- Such operators often come with some invariance under dilations: e.g.

$$\Delta D_t f = t^2 D_t \Delta f,$$

$$(D_t f) * |x|^{-(n-2)} = t^{-2} D_t (f * |x|^{-(n-2)}).$$

- Operators that exhibit modulation invariance (on top of translation and dilation invariances) are harder to analyze; they typically require rather refined *time-frequency analysis*.
- An example of such an operator is the Carleson operator, studied in connection with pointwise a.e. convergence of Fourier series of a function on $L^2(\mathbb{T})$:

$$\mathcal{C}f(x) = \sup_{N \in \mathbb{Z}} \left| \sum_{n \geq N} \widehat{f}(n) e^{2\pi i n x} \right|, \quad x \in \mathbb{T};$$

  note that $\mathcal{C}$ commutes with both translations and modulations, i.e.

$$\mathcal{C}\tau_y = \tau_y \mathcal{C} \quad \text{for all } y \in \mathbb{T}, \text{ and}$$

$$\mathcal{C}\Lambda_k = \Lambda_k \mathcal{C} \quad \text{for all } k \in \mathbb{Z}.$$

- Most of the operators we will encounter in this course will *not* be modulation invariant.