Topics in Harmonic Analysis
Lecture 2: Tempered distributions and Harmonic Functions

Po-Lam Yung

The Chinese University of Hong Kong

Introduction

- Last time we studied the Fourier transform of L^1 and L^2 functions.
- This time we extend this study to the more general context of *tempered distributions*.
- This will give us tools towards the study of some constant coefficient partial differential equations such as the Laplace equation.
- We will then be led to an introduction of many objects of interest in harmonic analysis, including the Riesz potentials, the Riesz transform and the Hilbert transform.

Outline

- Schwartz functions and tempered distributions
- The Laplace equation on $\mathbb{R}^n$
- Riesz potentials, Riesz transforms and Hilbert transform
- Harmonic functions on the upper half space
- Conjugate harmonic functions

Schwartz functions and tempered distributions

- The Schwartz space on $\mathbb{R}^n$ is defined as

$$\mathcal{S}(\mathbb{R}^n) := \{f \in C^\infty(\mathbb{R}^n) \colon \|f\|_{\alpha,\beta} < \infty \quad \text{for all multiindices } \alpha, \beta\},$$

  where

$$\|f\|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f(x)|$$

  for any multiindices $\alpha, \beta \in (\mathbb{N} \cup \{0\})^n$.

- It is a topological vector space, with $f_n \to 0$ in $\mathcal{S}(\mathbb{R}^n)$ if and only if

$$\|f_n\|_{\alpha,\beta} \to 0 \quad \text{for all multiindices } \alpha, \beta.$$

- It can be made a complete metric space, with

$$d(f,g) := \sum_{k=0}^{\infty} 2^{-k} \frac{\|f-g\|_k}{1 + \|f-g\|_k}$$

  where $\|f\|_k := \sup_{|\alpha|+|\beta|=k} \|f\|_{\alpha,\beta}$.

- The space of tempered distributions, denoted $\mathcal{S}'(\mathbb{R}^n)$, is the space of all continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$.
- Examples of tempered distributions include L^p functions for all $p \in [1, \infty]$, and locally integrable functions that grows at most polynomially at infinity (if u is such a function, then

$$u(f) := \int_{\mathbb{R}^n} u(x) f(x) dx, \quad f \in \mathcal{S}(\mathbb{R}^n)$$

  is a tempered distribution).
- Other examples include the Dirac delta functions δ_x for $x \in \mathbb{R}^n$:

$$\delta_x(f) := f(x), \quad f \in \mathcal{S}(\mathbb{R}^n)$$

  or more generally finite Borel measures with compact supports on $\mathbb{R}^n$.
- We often write $\langle u, f \rangle$ in lieu of $u(f)$ if $u \in \mathcal{S}'(\mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$.

- $\mathcal{S}'(\mathbb{R}^n)$ is a topological vector space with the weak* topology, i.e. $u_n \to 0$ in $\mathcal{S}'(\mathbb{R}^n)$ if and only if

$$\langle u_n, f\rangle \to 0 \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^n).$$

- A nice feature of $\mathcal{S}'(\mathbb{R}^n)$ is that one can take (distributional) derivative of any element u of $\mathcal{S}'(\mathbb{R}^n)$:

$$\langle \partial_j u, f\rangle := -\langle u, \partial_j f\rangle, \quad f \in \mathcal{S}(\mathbb{R}^n), \quad 1 \leq j \leq n.$$

  (Why is this an element of $\mathcal{S}'(\mathbb{R}^n)$?)

- The above definition of $\partial_j u$ is motivated by the identity

$$\int_{\mathbb{R}^n} (\partial_j u)(x) f(x) dx = -\int_{\mathbb{R}^n} u(x)(\partial_j f)(x) dx \quad \text{for all } u, f \in \mathcal{S}(\mathbb{R}^n)$$

  (and indeed the distributional derivative agrees with the classical derivative if $u \in C^1$).

- Similarly one can define translations, rotations and dilations of tempered distributions on $\mathbb{R}^n$:

$$\langle \tau_h u, f\rangle := \langle u, \tau_{-h} f\rangle, \quad f \in \mathcal{S}(\mathbb{R}^n), \quad h \in \mathbb{R}^n;$$

$$\langle u \circ R, f\rangle := \langle u, f \circ R^{-1}\rangle, \quad f \in \mathcal{S}(\mathbb{R}^n), \quad R \in O(n);$$

$$\langle D_\lambda u, f\rangle := \lambda^{-n}\langle u, D_{\lambda^{-1}} f\rangle, \quad f \in \mathcal{S}(\mathbb{R}^n), \quad \lambda > 0.$$

- A tempered distribution u is said to be homogeneous of degree α on $\mathbb{R}^n$ if

$$D_\lambda u = \lambda^\alpha u \quad \text{for all } \lambda > 0;$$

e.g. δ_0 is homogeneous of degree $-n$, and $|x|^\alpha$ is a homogeneous tempered distribution of degree α if $\alpha > -n$.

- One can multiply a tempered distribution u by a C^∞ function p that grows at most polynomially at infinity:

$$\langle pu, f\rangle := \langle u, pf\rangle, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

- In particular, one can multiply a tempered distribution by a polynomial, and modulate a tempered distribution:

$$\langle \Lambda_\xi u, f\rangle := \langle u, e^{2\pi i x\cdot\xi} f\rangle, \quad f \in \mathcal{S}(\mathbb{R}^n), \quad \xi \in \mathbb{R}^n.$$

- The Fourier transform defines a bijection on $\mathcal{S}(\mathbb{R}^n)$, and the Fourier inversion formula holds at every point:

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$$

whenever $f \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$; we also have the duality

$$\int_{\mathbb{R}^n} g(x)\widehat{f}(x)dx = \int_{\mathbb{R}^n} \widehat{g}(x)f(x)dx \quad \text{for all } f, g \in \mathcal{S}(\mathbb{R}^n).$$

- As a result, by duality one can define the Fourier transform of a tempered distribution u on $\mathbb{R}^n$:

$$\langle \widehat{u}, f \rangle := \langle u, \widehat{f} \rangle, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

- We sometimes also write $\mathcal{F}u$ for the Fourier transform of u if $u \in \mathcal{S}'(\mathbb{R}^n)$.

- e.g. $\widehat{\delta_x}$ is a function, given by $e^{-2\pi i x\cdot\xi}$; also

$$\mathcal{F}[(2\pi|x|)^{-\alpha}] = 2^\alpha \pi^{n/2} \frac{\Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)} |\xi|^{-(n-\alpha)} \quad \text{for } 0 < \alpha < n.$$

- If u is a homogeneous tempered distribution of degree α, then $\widehat{u}$ is a homogeneous tempered distribution of degree $-n-\alpha$.
- The Fourier transform defines a bijection on $\mathcal{S}'(\mathbb{R}^n)$. Thus one can take the inverse Fourier transform of any tempered distributions on $\mathbb{R}^n$.
- We sometimes also write $\mathcal{F}^{-1}u$ for the inverse Fourier transform of u if $u \in \mathcal{S}'(\mathbb{R}^n)$.

- If $g \in \mathcal{S}(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$, we can define the convolution of g with u, by

$$\langle u * g, f \rangle := \langle u, \tilde{g} * f \rangle, \quad f \in \mathcal{S}(\mathbb{R}^n)$$

  where $\tilde{g}(x) := g(-x)$. This works because $\tilde{g} * f$ is also Schwartz when both $f, g \in \mathcal{S}(\mathbb{R}^n)$.

- Indeed if $g \in \mathcal{S}(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$, then $u * g$ agrees with a C^∞ function on $\mathbb{R}^n$, given by

$$(u * g)(x) = \langle u, \tau_x \tilde{g} \rangle, \quad x \in \mathbb{R}^n.$$

- Furthermore, if $g \in \mathcal{S}(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$, then

$$\widehat{u * g} = \widehat{g} \cdot \widehat{u}.$$

The Laplace equation on $\mathbb{R}^n$

- We use the above technology of tempered distribution and Fourier transform to study the equation $-\Delta u = f$, where Δ is the Laplacian on $\mathbb{R}^n$ given by

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}.$$

- The equation $-\Delta u = f$ is equivalent, via the Fourier transform, to the equation $4\pi^2|\xi|^2\widehat{u}(\xi) = \widehat{f}(\xi)$.

- For $n \geq 2$, let m be a tempered distribution on $\mathbb{R}^n$ such that

$$\langle m, G \rangle = \int_{\mathbb{R}^n} \frac{G(\xi)}{(2\pi|\xi|)^2} d\xi$$

for every $G \in \mathcal{S}(\mathbb{R}^n)$ with $G(0) = 0$.

- An explicit such m is given by

$$\langle m, G \rangle := \begin{cases} \int_{\mathbb{R}^n} \frac{G(\xi)}{(2\pi|\xi|)^2} d\xi & \text{if } n \geq 3, \\ \int_{|\xi|\geq 1} \frac{G(\xi)}{(2\pi|\xi|)^2} d\xi + \int_{|\xi|\leq 1} \frac{G(\xi)-G(0)}{(2\pi|\xi|)^2} d\xi & \text{if } n = 2. \end{cases}$$

- Such m is not unique; one can for instance always add a multiple of the δ function at 0.
- We fix any such m in the next slide.

- If $f \in \mathcal{S}(\mathbb{R}^n)$ and $n \geq 2$, then

$$u := \mathcal{F}^{-1}(m\widehat{f}) \in \mathcal{S}'(\mathbb{R}^n)$$

  solves $-\Delta u = f$ in the sense of tempered distributions.

- This is because then for any $g \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\begin{aligned}
\langle 4\pi^2|\xi|^2\widehat{u}, g\rangle &= \langle \widehat{u}, (2\pi|\xi|)^2 g\rangle \\
&= \langle m, (2\pi|\xi|)^2\widehat{f}(\xi)g(\xi)\rangle \\
&= \int_{\mathbb{R}^n} \frac{(2\pi|\xi|)^2\widehat{f}(\xi)g(\xi)}{(2\pi|\xi|)^2} d\xi \\
&= \langle \widehat{f}, g\rangle
\end{aligned}$$

  (the second-to-last equality uses $(2\pi|\xi|)^2\widehat{f}(\xi)g(\xi) \in \mathcal{S}(\mathbb{R}^n)$ and vanishes at 0), which shows that

$$4\pi^2|\xi|^2\widehat{u} = \widehat{f}.$$

- This shows that if $f \in \mathcal{S}(\mathbb{R}^n)$ and $n \geq 2$, then

$$u := f * (\mathcal{F}^{-1} m) \in \mathcal{S}'(\mathbb{R}^n)$$

  solves $-\Delta u = f$ in the sense of tempered distributions.
- The above worked for a class of multipliers m; we may choose m suitably so that we have

$$\mathcal{F}^{-1} m = \begin{cases} -\frac{1}{2\pi} \log |x| & \text{if } n = 2 \\ \frac{\Gamma((n-2)/2)}{4\pi^{n/2}} |x|^{-(n-2)} & \text{if } n \geq 3. \end{cases}$$

  (Actually just take m as explicitly displayed two slides ago, and add a suitable multiple of δ_0 when $n = 2$.)
  These are called the fundamental solution to $-\Delta$ on $\mathbb{R}^n$.
- Thus we obtain a formula for a solution to $-\Delta u = f$ in the sense of tempered distributions if $f \in \mathcal{S}(\mathbb{R}^n)$ and $n \geq 2$.
- On the other hand, note that the solution to $-\Delta u = f$ is not unique in $\mathcal{S}'(\mathbb{R}^n)$: if u is a solution, then so is $u + P$ for any harmonic polynomial P on $\mathbb{R}^n$ (of which there are plenty).

Riesz potentials, Riesz transforms and Hilbert transform

- Motivated by the previous computation, given a tempered distribution $m \in \mathcal{S}'(\mathbb{R}^n)$, we consider operators of the form

$$f \in \mathcal{S}(\mathbb{R}^n) \mapsto \mathcal{F}^{-1}(m\widehat{f}) \in \mathcal{S}'(\mathbb{R}^n).$$

  This is called the *multiplier operator* associated to m.
- We specialize to the case when $m(\xi)$ agrees with the locally integrable function $(2\pi|\xi|)^{-\alpha}$ on $\mathbb{R}^n$ where $\alpha \in (0, n)$.
- We define the *Riesz potential* of order α on $\mathbb{R}^n$ by

$$\mathcal{I}_\alpha f = \mathcal{F}^{-1}((2\pi|\xi|)^{-\alpha}\widehat{f}(\xi))$$

  if $f \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha \in (0, n)$. $\mathcal{I}_\alpha$ is also sometimes written suggestively as $(-\Delta)^{-\alpha/2}$.
- The distributional solution to $-\Delta u = f$ we constructed on $\mathbb{R}^n$ in the last section is then just $\mathcal{I}_2 f = (-\Delta)^{-1} f$ if $n \geq 3$.

- By taking the inverse Fourier transform of $(2\pi|\xi|)^{-\alpha}$ on $\mathbb{R}^n$ when $\alpha \in (0, n)$, we see that

$$\mathcal{I}_\alpha f = f * \frac{\Gamma((n-\alpha)/2)}{2^\alpha \pi^{n/2} \Gamma(\alpha/2)} \frac{1}{|x|^{n-\alpha}}$$

  for all $f \in \mathcal{S}(\mathbb{R}^n)$.
- The mapping properties of $\mathcal{I}_\alpha$ on $\mathbb{R}^n$, for $\alpha \in (0, n)$, will be studied in the next lecture.

- Now we move on to the Riesz transforms on $\mathbb{R}^n$.
- Let $n \geq 2$ and $f \in \mathcal{S}(\mathbb{R}^n)$.
- Recall the distributional solution to $-\Delta u = f$ given by the fundamental solution of $-\Delta$.
- If $1 \leq j, k \leq n$, then

$$\widehat{\partial_{x_j}\partial_{x_k} u} = -\frac{\xi_j}{|\xi|}\frac{\xi_k}{|\xi|}\widehat{f}.$$

- This motivates us to study on $\mathbb{R}^n$ the multiplier operators with multiplier $-i\frac{\xi_j}{|\xi|}$, for $1 \leq j \leq n$.

- Such are called the *Riesz transforms* on $\mathbb{R}^n$: the j-th Riesz transform on $\mathbb{R}^n$ is given by

$$\widehat{R_j f}(\xi) = -i \frac{\xi_j}{|\xi|} \widehat{f}(\xi)$$

for $f \in \mathcal{S}(\mathbb{R}^n)$ and $1 \leq j \leq n$, if $n \geq 2$. Equivalently,

$$R_j f = f * \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \mathrm{p.v.} \frac{x_j}{|x|^{n+1}}.$$

- Thus if $f \in \mathcal{S}(\mathbb{R}^n)$ and u is the distributional solution to $-\Delta u = f$ given by the fundamental solution of $-\Delta$, then

$$\partial_{x_j} \partial_{x_k} u = R_j R_k f.$$

- The study of Riesz transforms on $\mathbb{R}^n$ is reduced to the study of the multiplier $i\mathrm{sgn}(\xi)$ on $\mathbb{R}$ if $n = 1$.
- This is called the Hilbert transform H on $\mathbb{R}$. More concretely,

$$\widehat{Hf}(\xi) = -i\mathrm{sgn}(\xi)\widehat{f}$$

if $f \in \mathcal{S}(\mathbb{R})$, or equivalently

$$Hf = f * \frac{1}{\pi}\mathrm{p.v.}\frac{1}{x}.$$

- The multiplier $-i\operatorname{sgn}(\xi)$ is, up to scalar multiplication, the unique homogeneous tempered distribution of degree 0 that is odd (i.e. that satisfies $m \circ R = -m$ where R is the reflection about the origin, given by $Rx = -x$).
- One can then give a characterization of the Hilbert transform along this line, using invariance properties.
- Beginning the next slide, we will see how the Hilbert transform also arises naturally from the study of conjugate harmonic functions on the upper half space $\mathbb{R}^2_+$.
- A similar discussion will be given for the Riesz transforms on $\mathbb{R}^n$, $n \geq 2$.
- The mapping properties of the Hilbert transform (on $\mathbb{R}$) and the Riesz transforms (on $\mathbb{R}^n$) will be deferred to Lecture 4.

Harmonic functions on the upper half space

- $\mathbb{R}$ arises as the boundary of the upper half space

$$\mathbb{R}^2_+ := \{(x, y) \in \mathbb{R}^2 \colon y > 0\}.$$

- Given $f \in \mathcal{S}(\mathbb{R})$, one may try to find a harmonic extension u of f into the upper half space $\mathbb{R}^2_+$, namely a continuous function u on the closure $\overline{\mathbb{R}^2_+}$ that solves the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{on } \mathbb{R}^2_+ \\ u = f & \text{on } \mathbb{R}. \end{cases}$$

- By taking partial Fourier transform in x and solving an ordinary differential equation in y, we are led to one candidate for such a function u, namely

$$u(x, y) = \int_{\mathbb{R}} \widehat{f}(\xi) e^{-2\pi y|\xi|} e^{2\pi i x\xi} d\xi.$$

- One easily verifies that then u is the Poisson integral of f, given by

$$u(x,y) = f * P_y(x), \quad y > 0$$

where

$$P_y(x) = \frac{1}{\pi} \frac{y}{x^2 + y^2}$$

is the Poisson kernel on $\mathbb{R}$.

- Hence one also sees that then u solves the Dirichlet problem on the last slide (since $P_y(x)$ forms an approximate identity).
- In addition, such $u(x,y)$ is bounded on $\mathbb{R}^2_+$, and is the unique bounded solution of the aforementioned Dirichlet problem.
- We now determine a suitable conjugate harmonic function v to u on $\mathbb{R}^2_+$, and consider its boundary value as $y \to 0^+$.

Conjugate harmonic functions

- Let $f \in \mathcal{S}(\mathbb{R})$ and

$$u(x,y) = \int_{\mathbb{R}} \widehat{f}(\xi) e^{-2\pi y|\xi|} e^{2\pi i x\xi} d\xi$$

  be the Poisson integral of f.

- We seek a harmonic conjugate of v, i.e. a harmonic function $v(x,y)$ on the upper half space $\mathbb{R}^2_+$ that satisfies

$$\partial_x v = \partial_y u, \quad \partial_y v = -\partial_x u.$$

- Taking partial Fourier transform in x, we see that one such v is given by

$$v(x,y) = -i \int_{\mathbb{R}} \mathrm{sgn}(\xi) \widehat{f}(\xi) e^{-2\pi y|\xi|} e^{2\pi i x\xi} d\xi;$$

  this is indeed the unique such v that is bounded on $\mathbb{R}^2_+$.

$$v(x,y) = -i \int_{\mathbb{R}} \mathrm{sgn}(\xi)\widehat{f}(\xi)e^{-2\pi y|\xi|}e^{2\pi ix\xi}d\xi;$$

- $v(x,y)$ can also be written as $v(x,y) = f * \tilde{P}_y(x)$ where

$$\tilde{P}_y(x) = \frac{1}{\pi}\frac{x}{x^2+y^2}$$

  is the conjugate Poisson kernel.
- We study the boundary behaviour of $v(x,y)$ as $y \to 0^+$.
- Since $f \in \mathcal{S}(\mathbb{R})$, the dominated convergence theorem shows that

$$\lim_{y\to 0^+} v(x,y) = -i \int_{\mathbb{R}} \mathrm{sgn}(\xi)\widehat{f}(\xi)e^{2\pi ix\xi}d\xi$$

  for every $x \in \mathbb{R}$.
- The right hand side is precisely the Hilbert transform of f.

- Alternatively, for $f \in \mathcal{S}(\mathbb{R})$, we also have

$$\lim_{y \to 0^+} v(x,y) = f * \frac{1}{\pi} \mathrm{p.v.} \frac{1}{x}$$

  since

$$\tilde{P}_y(x) \to \frac{1}{\pi} \mathrm{p.v.} \frac{1}{x}$$

  in $\mathcal{S}'(\mathbb{R})$ as $y \to 0^+$. Again we recognize the Hilbert transform of f in the limit.
- Hence if $f \in \mathcal{S}(\mathbb{R})$, then the Hilbert transform of f is the boundary value of a harmonic conjugate of the Poisson integral of f.

- A similar characterization can be given for the Riesz transforms in higher dimensions.
- Let $\mathbb{R}^{n+1}_+ = \{(x_0, x_1, \dots, x_n)\colon x_0 > 0\}$.
- For $f \in \mathcal{S}(\mathbb{R}^n)$, the Poisson integral of f is given by

$$\int_{\mathbb{R}^n} \widehat{f}(\xi) e^{-2\pi x_0 |\xi|} e^{2\pi i x\cdot\xi} d\xi = f * P_{x_0}(x)$$

where $P_{x_0}(x)$ is the Poisson kernel on $\mathbb{R}^{n+1}_+$ given by

$$P_{x_0}(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{x_0}{(x_0^2 + |x|^2)^{(n+1)/2}}.$$

This Poisson integral is the unique bounded solution of the Dirichlet problem on the upper half space $\mathbb{R}^{n+1}_+$:

$$\begin{cases} \Delta u = 0 & \text{on } \mathbb{R}^{n+1}_+ \\ u = f & \text{on } \mathbb{R}^n. \end{cases}$$

- A vector-valued function $(U_0, \ldots, U_n)$ defined on $\mathbb{R}^{n+1}_+$ is said to be a system of conjugate harmonic functions, if

$$\begin{cases} \partial_{x_j} U_k = \partial_{x_k} U_j \quad \text{for all } 0 \le j < k \le n \\ \sum_{j=0}^n \partial_{x_j} U_j = 0. \end{cases}$$

(If we let U be the differential 1-form on $\mathbb{R}^{n+1}_+$ given by $U = \sum_{j=0}^n U_j dx^j$, then the above system is equivalent to saying that U is a harmonic form on $\mathbb{R}^{n+1}_+$, i.e.

$$dU = 0 \quad \text{and} \quad d^* U = 0$$

where d and d^* are the divergence and curl operators respectively. Note that when $n = 1$ the above system is just the Cauchy-Riemann equations.)

- Now let $f \in \mathcal{S}(\mathbb{R}^n)$ and U_0 be the Poisson integral of f defined on $\mathbb{R}^{n+1}_+$.
- We then observe that $(U_0, \ldots, U_n)$ form a system of conjugate harmonic functions on $\mathbb{R}^{n+1}_+$ if

$$U_j(x_0, x) = -i \int_{\mathbb{R}^n} \frac{\xi_j}{|\xi|} \widehat{f}(\xi) e^{-2\pi x_0 |\xi|} e^{2\pi i x \cdot \xi} d\xi$$

  for $1 \leq j \leq n$.
- The boundary values of such are given by

$$\lim_{x_0 \to 0^+} U_j(x_0, x) = -i \int_{\mathbb{R}^n} \frac{\xi_j}{|\xi|} \widehat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi,$$

  which we recognize to be the j-th Riesz transform $R_j f$ of f.

- Alternatively, the U_j on the previous slide can be written as $U_j(x) = f * \tilde{P}_{x_0}^{(j)}(x)$ where

$$\tilde{P}_{x_0}^{(j)}(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{x_j}{(x_0^2 + |x|^2)^{(n+1)/2}}$$

is the j-th conjugate Poisson kernel for $1 \le j \le n$.

- The boundary values of such are given by

$$\lim_{x_0 \to 0^+} U_j(x_0, x) = f * \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \mathrm{p.v.} \frac{x_j}{|x|^{n+1}},$$

which we recognize again to be the j-th Riesz transform $R_j f$ of f, since

$$\tilde{P}_{x_0}^{(j)}(x) \to \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \mathrm{p.v.} \frac{x_j}{|x|^{n+1}}$$

in $\mathcal{S}'(\mathbb{R}^n)$ as $x_0 \to 0^+$.