Topics in Harmonic Analysis Lecture 3: Maximal functions and Riesz potentials

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Introduction

- Last time we saw some operators of interest in harmonic analysis, such as the Riesz potentials.
- We will study the Riesz potentials in more detail this time.
- Before that, we detour into a study of the Hardy-Littlewood maximal operator, whose study was motivated by another important question. We briefly describe this question next.
- The fundamental theorem of calculus says that if f is continuous at x, then

$$\frac{d}{dx}\int_0^x f(t)dt = f(x).$$

- In particular, if f is continuous at x, then

$$\lim_{r\to 0^+} \frac{1}{2r}\int_{(x-r,x+r)} f(t)dt = f(x).$$

$$\lim_{r \to 0^+} \frac{1}{2r} \int_{(x-r,x+r)} f(t)dt = f(x).$$

- We seek a variant of this, where we do not assume continuity of f at x.
- This variant will also extend to higher dimensions.
- The key issue here is the behaviour of averages of a locally integrable function f over balls of varying radii.
- For this reason we will study the Hardy-Littlewood maximal operator; what we gather will also ultimately enable us to come back and study some mapping properties of the Riesz potentials.

Outline

- L^p and weak-L^p spaces
- The Hardy-Littlewood maximal function
- Lebesgue differentiation theorem
- Boundary behaviour of Poisson integral
- Mapping properties of the Riesz potentials

L^p and weak L^p spaces

- The L^p space on $\mathbb{R}^n$ is the space of measurable functions on $\mathbb{R}^n$ for which $\|f\|_{L^p} < \infty$, where

$$\|f\|_{L^p} := \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p} \quad \text{when } 1 \leq p < \infty, \text{ and}$$

$$\|f\|_{L^\infty} := \inf\{M > 0 \colon |f(x)| \leq M \text{ for a.e. } x \in \mathbb{R}^n\}.$$

- By Fubini's theorem, we have

$$\|f\|_{L^p} = \left(\int_0^\infty p\alpha^{p-1} |\{x \in \mathbb{R}^n \colon |f(x)| > \alpha\}| d\alpha \right)^{1/p}$$

  for $1 \leq p < \infty$.
- The function $\alpha \mapsto |\{x \in \mathbb{R}^n \colon |f(x)| > \alpha\}|$ is sometimes called the distribution function of f.
- The Chebyshev's inequality says that if $f \in L^p$, then

$$|\{x \in \mathbb{R}^n \colon |f(x)| > \alpha\}| \leq \frac{1}{\alpha^p} \|f\|_{L^p}^p \quad \text{for all } \alpha > 0.$$

- For $1 \leq p < \infty$, the weak L^p space on $\mathbb{R}^n$ (denoted $L^{p,\infty}$) is the space of measurable functions f on $\mathbb{R}^n$ for which there exists a constant C such that

$$|\{x \in \mathbb{R}^n \colon |f(x)| > \alpha\}| \leq \frac{C^p}{\alpha^p} \quad \text{for all } \alpha > 0.$$

- The smallest constant C for which the above inequality holds for all $\alpha > 0$ is precisely

$$\sup_{\alpha > 0} \left[\alpha |\{x \in \mathbb{R}^n \colon |f(x)| > \alpha\}|^{1/p} \right].$$

  Hence f is in $L^{p,\infty}$, if and only if the above supremum is finite.
- By Chebyshev, L^p embeds into $L^{p,\infty}$ for $1 \leq p < \infty$, but the embedding is strict.
- e.g. $|x|^{-n/p} \in L^{p,\infty}$ for all $1 \leq p < \infty$ (but not in L^p).

- $L^p(\mathbb{R}^n)$ is a Banach space for all $1 \le p \le \infty$.
- On the other hand, for $1 \le p < \infty$, the supremum defining $L^{p,\infty}$ on the last slide, namely

$$\sup_{\alpha>0} \left[\alpha |\{x \in \mathbb{R}^n \colon |f(x)| > \alpha\}|^{1/p}\right]$$

defines only a quasi-norm but not a norm; it only satisfies a quasi-triangle inequality, but not the triangle inequality itself.
- Nevertheless, when $1 < p < \infty$, there is a comparable quantity

$$\|f\|_{L^{p,\infty}} := \sup_{\substack{E \text{ measurable} \\ 0<|E|<\infty}} \frac{1}{|E|^{1/p'}} \int_{\mathbb{R}^n} |f| \chi_E dx$$

which is a norm on $L^{p,\infty}$ and turn $L^{p,\infty}$ into a Banach space (indeed this identifies $L^{p,\infty}$ as the dual of another Banach space $L^{p',1}$ when $1 < p < \infty$).

The Hardy-Littlewood maximal function

- Let f be a locally integrable function on $\mathbb{R}^n$.
- Write $B(x,r)$ for the ball of radius r centered at x.
- Define the Hardy-Littlewood maximal operator by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(t)|dt \qquad \text{for every } x \in \mathbb{R}^n.$$

- It is the maximal average of $|f|$ over all balls centered at x.
- Note that M is a sublinear operator:

$$M(f+g) \leq Mf + Mg.$$

  Mf is also lower semi-continuous for every f: the set $\{x \in \mathbb{R}^n \colon Mf(x) > \alpha\}$ is open for every $\alpha \in \mathbb{R}$.
- We are interested in the mapping properties of M on L^p or weak L^p.

- Indeed we will show that M is bounded on L^p for all $1 < p \leq \infty$.
- It is easy to see that M is not bounded on L^1; indeed $Mf \notin L^1$ unless $f = 0$ a.e.
- Nevertheless, a substitute result is available for the action of M on L^1.
- We will show that M maps L^1 boundedly into weak-L^1, and that's the key to the proof of the boundedness of M on L^p $(1 < p < \infty)$ as well.
- The key then is to interpolate the fact that $M\colon L^1 \to L^{1,\infty}$ with the easy observation that $M\colon L^\infty \to L^\infty$.
- Terminology: a sublinear operator is said to be of strong-type (p, q) if it defines a bounded operator from L^p into L^q; and it is said to be of weak-type (p, q) if it defines a bounded operator from L^p into weak-L^q.

Theorem

M is of weak-type $(1,1)$ on $\mathbb{R}^n$, i.e. there exists a constant $C_n > 0$ such that for any $\alpha > 0$,

$$|\{x \in \mathbb{R}^n \colon |Mf(x)| > \alpha\}| \leq \frac{C_n}{\alpha} \|f\|_{L^1(\mathbb{R}^n)}$$

- The proof proceeds via the following covering lemma:

Lemma

Let $E \subset \mathbb{R}^n$, and suppose there exists a finite collection of open balls $\mathcal{B}$ that covers E. Then there exists a subcollection $B_1, \ldots, B_N \in \mathcal{B}$ such that

- *$B_1, \ldots, B_N$ are pairwise disjoint; and*
- *$3B_1, \ldots, 3B_N$ covers E, where $3B_j$ is the ball with the same center as B_j but three times the radius.*

- Assume the lemma for now. We will prove the theorem.
- Let $f \in L^1(\mathbb{R}^n)$, and $\alpha > 0$. Let E_α be any compact subset of the open set $\{x \in \mathbb{R}^n \colon |Mf(x)| > \alpha\}$.
- By inner regularity of the Lebesgue measure, it suffices to prove that
$$|E_\alpha| \leq \frac{C_n}{\alpha} \|f\|_{L^1(\mathbb{R}^n)}$$
with a constant independent of E_α.
- Now for each $x \in E_\alpha$, there exists some radius $r_x > 0$ such that
$$\frac{1}{|B(x, r_x)|} \int_{B(x,r_x)} |f| > \alpha.$$
- The collection of open balls $\{B(x, r_x) \colon x \in E_\alpha\}$ covers E_α, and since E_α is compact, we can select a finite subcover $\mathcal{B}_\alpha$ of E_α from this collection.

- Now apply the covering lemma to E_α and this collection of balls $\mathcal{B}_\alpha$.
- We obtain a subcollection $B_1, \dots, B_N \in \mathcal{B}_\alpha$ such that $B_1, \dots, B_N$ are pairwise disjoint, and $E_\alpha \subset \bigcup_{j=1}^N 3B_j$.
- As a result,

$$|E_\alpha| \leq \sum_{j=1}^N |3B_j| = 3^n \sum_{j=1}^N |B_j| \leq \frac{3^n}{\alpha} \sum_{j=1}^N \int_{B_j} |f| \leq \frac{3^n}{\alpha} \|f\|_{L^1},$$

  the last inequality following since $B_1, \dots, B_N$ are pairwise disjoint.
- This proves the theorem with $C_n = 3^n$, modulo the proof of the covering lemma.
  (This constant is not sharp; one can replace it with $(2+\varepsilon)^n$ for any $\varepsilon > 0$ by using a more refined covering lemma.)

- The proof of the covering lemma is by greedy algorithm:
- Just let B_1 be a ball in $\mathcal{B}$ with maximal radius (possible since $\mathcal{B}$ is only a finite collection).
- Throw away all balls in $\mathcal{B}$ that intersects B_1, and let B_2 be a ball in the remaining collection whose radius is maximal.
- Repeat this process until no balls are left.
- The process will terminate since we have only a finite collection of balls.
- The chosen balls are clearly pairwise disjoint.
- Any ball that is thrown away intersects one of the chosen balls with a larger or equal radius.
- Thus any ball that is thrown away is contained in $3B_j$ for some chosen ball B_j, and this shows $3B_1, 3B_2, \dots$ cover E.
- This finishes the proof of the covering lemma.

- Next we prove the following theorem:

Theorem

M is of strong-type (p,p) on $\mathbb{R}^n$ for all $1 < p \leq \infty$, i.e. for any such p, there exists a constant $C_{n,p}$ such that

$$\|Mf\|_{L^p} \leq C_{n,p}\|f\|_{L^p}$$

for all $f \in L^p(\mathbb{R}^n)$.

- Since clearly $Mf(x) \leq \|f\|_{L^\infty}$ for every $x \in \mathbb{R}^n$, the theorem is trivial when $p = \infty$ (with $C_{n,\infty} = 1$).
- We will prove the theorem by interpolating this L^∞ endpoint with the weak-type (1,1) result we just proved.
- This gives a constant $C_{n,p}$ that depends on n.
- On the other hand, we remark that via more sophisticated methods, the constant $C_{n,p}$ can be chosen independent of n for all $1 < p \leq \infty$. We will not pursue this here.

- The starting point of the proof of the theorem is the following identity:

$$\|Mf\|_{L^p}^p = \int_0^\infty p\alpha^{p-1}|\{x \in \mathbb{R}^n \colon Mf(x) > \alpha\}|d\alpha$$

which holds for all $1 < p < \infty$ by Fubini's theorem.

- Now for each $\alpha > 0$, we have

$$f = f\chi_{|f|>\alpha/2} + f\chi_{|f|\leq\alpha/2}$$

and $M(f\chi_{|f|\leq\alpha/2})(x) \leq \alpha/2$ for every $x \in \mathbb{R}^n$, by the boundedness of M on L^∞.

- Thus by subadditivity,

$$\{x \in \mathbb{R}^n \colon Mf(x) > \alpha\} \subseteq \{x \in \mathbb{R}^n \colon M(f\chi_{|f|>\alpha/2})(x) > \alpha/2\}.$$

- Since M is of weak-type (1,1), the measure of the latter is at most

$$\frac{2C_n}{\alpha}\int_{\mathbb{R}^n} |f|\chi_{|f|>\alpha/2}.$$

- Thus

$$
\begin{aligned}
\|Mf\|_{L^p}^p &= \int_0^\infty p\alpha^{p-1}|\{x \in \mathbb{R}^n \colon Mf(x) > \alpha\}|d\alpha \\
&\leq \int_0^\infty p\alpha^{p-1}\frac{2C_n}{\alpha}\int_{\mathbb{R}^n}|f(x)|\chi_{|f|>\alpha/2}(x)dxd\alpha \\
&\leq 2C_n p\int_{\mathbb{R}^n}|f(x)|\int_0^{2|f(x)|}\alpha^{p-2}d\alpha dx \\
&\leq 2C_n\frac{p}{p-1}\int_{\mathbb{R}^n}2^{p-1}|f(x)|^p dx \\
&= C_n 2^p\frac{p}{p-1}\|f\|_{L^p}^p.
\end{aligned}
$$

- This proves the theorem with $C_{n,p} = 2(p')^{1/p}C_n^{1/p}$ where $p' = p/(p-1)$ is the Hölder conjugate of p.
- Note that this constant blows up like $O(1/(p-1))$ as $p \to 1^-$.
- The above method of proof is an example of the technique of real interpolation. We will return to this in Lecture 8.

Lebesgue differentiation theorem

- We may now prove the Lebesgue differentiation theorem, which can be thought of as a measure-theoretic version of the fundamental theorem of calculus in 1-dimension.

Theorem

Let f be a locally integrable function on $\mathbb{R}^n$. Then for a.e. $x \in \mathbb{R}^n$, we have

$$f(x) = \lim_{r \to 0^+} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(t)dt.$$

- Without loss of generality we assume that f is compactly supported (and hence in L^1).
- If f were also continuous, then the conclusion of the theorem clearly holds for all $x \in \mathbb{R}^n$.
- The idea is to approximate f in L^1 by a continuous function with compact support.

- Suppose $f \in L^1(\mathbb{R}^n)$. We will prove a slightly stronger statement:

$$\limsup_{r\to 0^+} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(t)-f(x)|dt = 0$$

for a.e. $x \in \mathbb{R}^n$.

- Let $\varepsilon > 0$. Let $g \in C_c(\mathbb{R}^n)$ be such that $\|f-g\|_{L^1(\mathbb{R}^n)} \leq \varepsilon$.
- Then

$$\begin{aligned}
&\limsup_{r\to 0^+} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(t)-f(x)|dt \\
\leq &\limsup_{r\to 0^+} \frac{1}{|B(x,r)|}\int_{B(x,r)} |g(t)-g(x)|dt \\
&+ \limsup_{r\to 0^+} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(t)-g(t)|dt + |f(x)-g(x)| \\
\leq &M|f-g|(x) + |f-g|(x)
\end{aligned}$$

- Hence for any $\alpha > 0$, we have

$$
\begin{aligned}
&|\{x \in \mathbb{R}^n : \limsup_{r \to 0^+} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(t) - f(x)| dt > \alpha\}| \\
\leq &|\{x \in \mathbb{R}^n : M|f-g|(x) > \alpha/2\}| + |\{x \in \mathbb{R}^n : |f-g|(x) > \alpha/2\}| \\
\leq &\frac{C_n}{\alpha/2} \|f-g\|_{L^1(\mathbb{R}^n)} \leq \frac{2C_n}{\alpha} \varepsilon.
\end{aligned}
$$

- Letting $\varepsilon \to 0$, we see that

$$
|\{x \in \mathbb{R}^n : \limsup_{r \to 0^+} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(t) - f(x)| dt > \alpha\}| = 0
$$

for all $\alpha > 0$, i.e.

$$
\limsup_{r \to 0^+} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(t) - f(x)| dt = 0
$$

for a.e. $x \in \mathbb{R}^n$.

- More generally, we have the following generalization of Lebesgue's differentiation theorem:

Theorem

Suppose $\phi \in L^1(\mathbb{R}^n)$ *with* $\int_{\mathbb{R}^n} \phi(x)dx = 1$*. Let* ψ *be the least radial decreasing majorant of* $|\phi|$*, i.e.*

$$\psi(x) = \sup_{|y| \geq |x|} |\phi(y)|.$$

Suppose $\psi \in L^1(\mathbb{R}^n)$*. Let* $\phi_r(x) = r^{-n}\phi(r^{-1}x)$ *for* $r > 0$*. Let* f *be an* L^p *function on* $\mathbb{R}^n$ *for some* $1 \leq p \leq \infty$*. Then we have*

$$\sup_{r>0} |f * \phi_r|(x) \leq A\, Mf(x)$$

for every $x \in \mathbb{R}^n$*, where* $A = \int_{\mathbb{R}^n} \psi(x)dx$*. Also,*

$$f(x) = \lim_{r \to 0^+} f * \phi_r(x) \quad \text{for a.e. } x \in \mathbb{R}^n.$$

- Indeed, let $\psi_r(x) = r^{-n}\psi(r^{-1}x)$. We claim that

$$\sup_{r>0} |f| * \psi_r(x) \leq A\, Mf(x)$$

for all $x \in \mathbb{R}^n$, where $A = \int_{\mathbb{R}^n} \psi(x)dx$.

- If this claim is verified, then at any x where $Mf(x) < \infty$, we have $|f| * \psi_r(x) < \infty$ for all $r > 0$, and hence the integral defining $f * \phi_r(x)$ converges for all $r > 0$.
- It then remains to observe that

$$\sup_{r>0} |f * \phi_r(x)| \leq \sup_{r>0} |f| * \psi_r(x) \leq A\, Mf(x)$$

which is the desired conclusion.

- The claim can be proved by approximating ψ from below by linear combinations of characteristic functions of balls centered at the origin.

- More precisely, one can find a sequence of functions $\{\rho_k\}_{k=1}^{\infty}$ increasing pointwisely to ψ, such that each ρ_k is a finite sum of the form

$$\rho_k(x) = \sum_j a_{j,k} \chi_{B_{j,k}}$$

for some non-negative coefficients $a_{j,k}$ and some balls $B_{j,k}$ centered at the origin, and such that

$$\sum_j a_{j,k} |B_{j,k}| \leq A \quad \text{for any } k \in \mathbb{N}.$$

- Then

$$|f| * \psi_r(x) = \lim_{k \to \infty} |f| * (\rho_k)_r(x),$$

where $(\rho_k)_r(x) = r^{-n} \rho_k(r^{-1}x)$, and

$$|f| * (\rho_k)_r(x) = \sum_j a_{j,k} |f| * (\chi_{B_{j,k}})_r(x)$$

$$\leq \sum_j a_{j,k} |B_{j,k}| Mf(x) \leq A\, Mf(x)$$

for any $k \in \mathbb{N}$, $r > 0$ and $x \in \mathbb{R}^n$.

- Once we established that $\sup_{r>0} |f * \phi_r(x)| \leq A\, Mf(x)$ for all $x \in \mathbb{R}^n$, then to prove

$$f(x) = \lim_{r \to 0^+} f * \phi_r(x)$$

for a.e. $x \in \mathbb{R}^n$, we may proceed as before when $1 \leq p < \infty$.

- We only need to note that the above identity holds for every $x \in \mathbb{R}^n$ if f were in addition continuous with compact support.
- To see the latter fact, let $f \in C_c(\mathbb{R}^n)$. Let $\varepsilon > 0$. Then we can choose $R > 0$ large enough, so that $\int_{|y| \geq R} |\phi(y)| dy < \varepsilon$. Then

$$f * \phi_r(x) - f(x) = \int_{\mathbb{R}^n} [f(x - ry) - f(x)]\phi(y) dy,$$

and we split this integral into two parts depending on whether $|y| \leq R$ or $|y| \geq R$.

- The integral over $|y| \geq R$ is bounded by $2\|f\|_{L^\infty}\varepsilon$.
- The integral over $|y| \leq R$ can be made smaller than ε if r were chosen small enough, by uniform continuity of f.

- Now suppose $f \in L^p(\mathbb{R}^n)$ with $1 \le p < \infty$.
- Let $\varepsilon > 0$. Let $g \in C_c(\mathbb{R}^n)$ be such that $\|f - g\|_{L^p(\mathbb{R}^n)} \le \varepsilon$.
- Then for any $\alpha > 0$, we have

$$\begin{aligned} &|\{x \in \mathbb{R}^n \colon \limsup_{r \to 0^+} |f * \phi_r(x) - f(x)| > \alpha\}| \\ \le &|\{x \in \mathbb{R}^n \colon 2M|f - g|(x) > \alpha\}| \\ \le &\frac{C_{n,p}}{\alpha^p} \|f - g\|_{L^p(\mathbb{R}^n)}^p \le \frac{C_{n,p}}{\alpha^p} \varepsilon^p. \end{aligned}$$

- Letting $\varepsilon \to 0$ we see that $f(x) = \lim_{r \to 0^+} f * \phi_r(x)$ for a.e. $x \in \mathbb{R}^n$.
- When $f \in L^\infty(\mathbb{R}^n)$, a small modification is necessary: we will instead prove that

$$f(x) = \lim_{r \to 0^+} f * \phi_r(x)$$

for a.e. $x \in B(0, R)$, for every $R > 0$.

- To do so, it suffices to let

$$f = f\chi_{B(0,2R)} + f\chi_{B(0,2R)^c} = f_1 + f_2,$$

and verify pointwise a.e. convergence in $B(0,R)$ for each of them.

- But for every $x \in B(0,R)$,

$$\begin{aligned} |f_2 * \phi_r(x)| &= \left| \int_{|y| \geq 2R} f(y)\phi_r(x-y)dy \right| \\ &\leq \|f\|_{L^\infty(\mathbb{R}^n)} \int_{|y| \geq R} \phi_r(y)dy \to 0 \end{aligned}$$

as $r \to 0^+$. Also since $f_1 \in L^1(\mathbb{R}^n)$, we have

$$\lim_{r\to 0^+} f_1 * \phi_r(x) = f_1(x) = f(x) \quad \text{for a.e. } x \in B(0,R).$$

Thus $\lim_{r\to 0^+} f * \phi_r(x) = f(x)$ for a.e. $x \in B(0,R)$.

Boundary behaviour of Poisson integral

- We have thus completed the proof of the generalization of the Lebesgue differentiation theorem.
- As an application, this allows us to study the behaviour of the Poisson integral $u(x, y)$ of a function $f(x)$ on $\mathbb{R}^n$ as $y \to 0^+$.

Theorem

Let $f(x)$ be an L^p function on $\mathbb{R}^n$ for some $1 \leq p \leq \infty$, and let $u(x, y) = f * P_y(x)$ be its Poisson integral for $(x, y) \in \mathbb{R}^{n+1}_+$. Then we have

$$\lim_{y \to 0^+} u(x, y) = f(x) \quad \textit{for a.e. } x \in \mathbb{R}^n.$$

In addition,

$$\sup_{y>0} |u(x, y)| \leq Mf(x) \quad \textit{for every } x \in \mathbb{R}^n.$$

- We remark that we also have L^p norm convergence of $u(x, y)$ to $f(x)$ if $f \in L^p(\mathbb{R}^n)$ and $p \in [1, \infty)$.

Mapping properties of Riesz potentials

- Finally, we establish mapping properties of the Riesz potentials on $L^p(\mathbb{R}^n)$ from the mapping properties of the Hardy-Littlewood maximal function.
- Recall the Riesz potentials $\mathcal{I}_\alpha \colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, defined by

  $$\mathcal{I}_\alpha f = (-\Delta)^{-\alpha/2} f = \mathcal{F}^{-1}((2\pi|\xi|)^{-\alpha}\widehat{f}(\xi)) = c_{n,\alpha} f * |x|^{-(n-\alpha)}$$

  for some explicit constant $c_{n,\alpha}$ if $\alpha \in (0,n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$.
- The kernel $|x|^{-(n-\alpha)}$ is in $L^{\frac{n}{n-\alpha},\infty}(\mathbb{R}^n)$ but not in $L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$.
- If it were in $L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$, then Young's convolution inequality says that $\mathcal{I}_\alpha \colon L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ where

  $$\frac{1}{q} = \frac{1}{p} + \frac{n-\alpha}{n} - 1 = \frac{1}{p} - \frac{\alpha}{n} \quad \text{whenever } 1 \le p \le n/\alpha.$$

- Remarkably, this mapping property remains true when $1 < p < n/\alpha$ even though $|x|^{-(n-\alpha)} \notin L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$.

Theorem

For $\alpha \in (0, n)$ and $1 \leq p < n/\alpha$, let

$$\frac{1}{p^*} = \frac{1}{p} - \frac{\alpha}{n}.$$

Then we have

(a) $\mathcal{I}_\alpha$ *is of weak type* $(1, 1^*)$ *on* $\mathbb{R}^n$*;*

(b) $\mathcal{I}_\alpha$ *is of strong type* (p, p^*) *on* $\mathbb{R}^n$ *if* $1 < p < n/\alpha$*.*

- To prove this, let $1 \leq p < n/\alpha$.
- Note that for each $x \in \mathbb{R}^n$, we have

$$\mathcal{I}_\alpha f(x) = c_{n,\alpha} \int_{y \in \mathbb{R}^n} f(x-y) \frac{1}{|y|^{n-\alpha}} dy.$$

- Since the kernel of $\mathcal{I}_\alpha$ is non-negative, we may assume that f is non-negative.
- We split the integral into two parts, depending on whether $|y| \leq R$ or $|y| \geq R$, where $R > 0$ is to be chosen.

$$\mathcal{I}_\alpha f(x) = c_{n,\alpha} \int_{y\in\mathbb{R}^n} f(x-y)\frac{1}{|y|^{n-\alpha}}dy = \int_{|y|\leq R} + \int_{|y|>R}$$

- We estimate the first integral by the Hardy-Littlewood maximal function: indeed, since $\int_{|y|\leq R} \frac{1}{|y|^{n-\alpha}}dy \leq C_{\alpha,n}R^\alpha$, by a previous theorem, we have

$$\int_{|y|\leq R} f(x-y)\frac{1}{|y|^{n-\alpha}}dy \leq C_{\alpha,n}R^\alpha Mf(x).$$

- We estimate the second integral by Hölder's inequality: indeed, since $p < n/\alpha$, we have $\chi_{|y|>R}|y|^{-(n-\alpha)} \in L^{p'}(\mathbb{R}^n)$, and hence

$$\int_{|y|>R} f(x-y)\frac{1}{|y|^{n-\alpha}}dy \leq C_{\alpha,n,p}R^{\alpha-\frac{n}{p}}\|f\|_{L^p}.$$

- Thus

$$\mathcal{I}_\alpha f(x) \lesssim_{\alpha,n,p} R^\alpha Mf(x) + R^{\alpha-\frac{n}{p}}\|f\|_{L^p}.$$

$$\mathcal{I}_\alpha f(x) \lesssim_{\alpha,n,p} R^\alpha Mf(x) + R^{\alpha-\frac{n}{p}}\|f\|_{L^p}$$

- We choose R so that the right hand side is almost minimized, say so that $R^\alpha Mf(x) = R^{\alpha-\frac{n}{p}}\|f\|_{L^p}$. Then

$$\mathcal{I}_\alpha f(x) \lesssim_{\alpha,n,p} \|f\|_{L^p}^{\frac{\alpha p}{n}} Mf(x)^{1-\frac{\alpha p}{n}} = \|f\|_{L^p}^{1-\frac{p}{p^*}} Mf(x)^{\frac{p}{p^*}}.$$

- This shows

$$\|\mathcal{I}_\alpha f(x)\|_{L^{p^*}} \lesssim_{\alpha,n,p} \|f\|_{L^p}^{1-\frac{p}{p^*}} \|Mf(x)\|_{L^p}^{\frac{p}{p^*}} \lesssim_{\alpha,n,p} \|f\|_{L^p}$$

if $1 < p < n/\alpha$, whereas for $p = 1$ we have

$$\begin{aligned} &|\{x \in \mathbb{R}^n \colon \mathcal{I}_\alpha f(x) > \alpha\}| \\ &\le |\{x \in \mathbb{R}^n \colon Mf(x) > C_{\alpha,n}\alpha^{1^*}\|f\|_{L^p}^{-1^*+1}\}| \\ &\lesssim_{\alpha,n} \alpha^{-1^*}\|f\|_{L^1}^{1^*-1}\|f\|_{L^1} = \alpha^{-1^*}\|f\|_{L^1}^{1^*}. \end{aligned}$$

- Hence $\mathcal{I}_\alpha$ is strong type (p, p^*) if $1 < p < n/\alpha$, and weak type $(1, 1^*)$, as desired.

- We remark that using the mapping properties of $\mathcal{I}_\alpha$ we just proved, and certain rearrangement arguments, one can establish the following generalized Young's convolution inequality on $\mathbb{R}^n$:

Theorem

Suppose $1 < p, q, r < \infty$ *and*

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

Then for any $f \in L^p(\mathbb{R}^n)$, $g \in L^{q,\infty}(\mathbb{R}^n)$, *we have*

$$\|f * g\|_{L^r(\mathbb{R}^n)} \lesssim_{p,q,r} \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^{q,\infty}(\mathbb{R}^n)}.$$

- Instead of giving this rearrangement argument here, we will give a proof of this theorem in Homework 8, as an application of a technique called real interpolation (which incidentally allows one to refine this inequality further).