Topics in Harmonic Analysis
Lecture 4: Singular integrals
and Littlewood-Paley decompositions

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Introduction

- Last time we studied mapping properties of maximal functions and the Riesz potentials.
- The latter involves (non-negative) integral kernels in a weak-L^q space for some $1 < q < \infty$, in lieu of the (strong) L^q.
- This time we study singular integrals, which are convolutions with certain (signed) integral kernels that belong to weak-L^1.
- Examples include the Hilbert transform and the Riesz transforms we have seen in Lecture 2; other multiplier operators will also be discussed.
- An important application will be given to the Littlewood-Paley decomposition of functions in L^p, $1 < p < \infty$.
- Note that when $p \neq 2$, L^p is not a Hilbert space, and hence the notion of orthogonality is not immediately present. The Littlewood-Paley decomposition often allows one to resurrect certain orthogonality in L^p spaces, and is hence very useful.

Outline

- Singular integral operators: an introduction
- The Calderón-Zygmund decomposition
- Mapping properties of singular integral operators
- Hörmander-Mikhlin multipliers
- A vector-valued version of the main theorem
- Littlewood-Paley decompositions

Singular integral operators: an introduction

- From Young's convolution inequality, we know that if $K \in L^1(\mathbb{R}^n)$, then the convolution operator

$$f \mapsto f * K$$

is bounded on $L^p(\mathbb{R}^n)$ for any $1 \leq p \leq \infty$.

- But many operators of interest in harmonic analysis involve convolution kernels that are not in L^1, that are only in $L^{1,\infty}$.
- Examples include the Hilbert transform on $\mathbb{R}$:

$$Hf = f * \frac{1}{\pi} \mathrm{p.v.} \frac{1}{x},$$

as well as the Riesz transforms on $\mathbb{R}^n$:

$$R_j f = f * \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \mathrm{p.v.} \frac{x_j}{|x|^{n+1}}.$$

- The Hilbert transform and the Riesz transforms will be prototypes of what we call singular integral operators on $\mathbb{R}^n$.
- We want to study mapping properties of such on $L^p(\mathbb{R}^n)$.
- Note that the convolution kernels of the earlier operators are not just in $L^{1,\infty}$; they satisfy certain *cancellation conditions*. This is important for what follows.
- In particular, the kernels of both the Hilbert transform and the Riesz transforms takes on both positive and negative values (indeed the kernels are odd).
- Things fail if we replace these convolution kernels with their non-negative counterparts.
- It is known, for instance, that $f \mapsto f * \frac{1}{|x|}$ (appropriately defined) is not bounded on $L^p(\mathbb{R})$ for any $1 \leq p \leq \infty$.
- Before we turn to the mapping properties of singular integral operators, we need to establish the important *Calderón-Zygmund decomposition* of an L^1 function.

The Calderón-Zygmund decomposition

Theorem

Let $f \in L^1(\mathbb{R}^n)$ and $\alpha > 0$. Then there exists a decomposition $f = g + b$, such that

$$\|g\|_{L^1} + \|b\|_{L^1} \leq C_n \|f\|_{L^1},$$

$$\|g\|_{L^\infty} \leq C_n \alpha,$$

In addition, b can be further decomposed into $b = \sum_j b_j$, so that each b_j is supported on a cube Q_j (with $0 < |Q_j| < \infty$),

$$\int_{Q_j} b_j(y) dy = 0 \quad \text{for all } j,$$

the Q_j's are essentially disjoint (in the sense that $|Q_j \cap Q_k| = 0$ whenever $j \neq k$), and that

$$\sum_j |Q_j| \leq \frac{C_n}{\alpha} \|f\|_{L^1}.$$

- To establish this Calderón-Zygmund decomposition, let $f \in L^1(\mathbb{R}^n)$ and $\alpha > 0$.
- Tile $\mathbb{R}^n$ by essentially disjoint cubes of side lengths L, where L is chosen so large so that $⨍_Q |f| < \alpha$ for each cube Q in the collection ($⨍_Q = \frac{1}{|Q|} \int_Q$).
- Subdivide each cube into 2^n cubes of equal sizes, and consider $⨍_{Q'} |f|$ for each smaller cube Q' that arises. If this average is $\geq \alpha$, we collect Q' into a collection $\mathcal{Q}$; if not, then we keep subdividing.
- We end up with a countable collection $\mathcal{Q}$ of essentially disjoint cubes so that for each $Q \in \mathcal{Q}$, we have

$$⨍_Q |f| \geq \alpha, \quad \text{whereas} \quad ⨍_{\tilde{Q}} |f| < \alpha$$

where $\tilde{Q}$ is the 'parent' of Q (the cube from which Q was obtained by subdivision).

- Thus we have obtained a countable collection $\mathcal{Q}$ of essentially disjoint cubes

$$\alpha \le \fint_Q |f| \le 2^n \alpha \quad \text{for all } Q \in \mathcal{Q}.$$

- We also note that if $\Omega = \bigcup_{Q \in \mathcal{Q}} Q$, then

$$|f(x)| \le \alpha \quad \text{for a.e. } x \notin \Omega$$

by the Lebesgue differentiation theorem.

- It suffices now to enumerate $\mathcal{Q}$ as $\{Q_1, Q_2, \dots\}$, and define

$$g(x) = \begin{cases} f(x) & \text{if } x \notin \Omega \\ \fint_{Q_j} f(y)dy & \text{if } x \in Q_j \text{ for some } j, \end{cases}$$

$$b_j(x) = \begin{cases} f(x) - \fint_{Q_j} f(y)dy & \text{if } x \in Q_j \\ 0 & \text{otherwise.} \end{cases}$$

for each j, and $b = \sum_j b_j$.

$$g(x) = \begin{cases} f(x) & \text{if } x \notin \Omega \\ \fint_{Q_j} f(y)dy & \text{if } x \in Q_j \text{ for some } j, \end{cases}$$

$$b_j(x) = \begin{cases} f(x) - \fint_{Q_j} f(y)dy & \text{if } x \in Q_j \\ 0 & \text{otherwise.} \end{cases}$$

- Indeed then

$$\|g\|_{L^1} \leq \|f\|_{L^1}, \quad \|g\|_{L^\infty} \leq 2^n \alpha,$$

$$\int_{Q_j} b_j = 0 \quad \text{and} \quad |Q_j| \leq \frac{1}{\alpha} \int_{Q_j} |f| \quad \text{for all } j,$$

from which all desired properties of g and b can be easily derived.

- Alternatively, we apply the following Whitney decomposition theorem for open sets in $\mathbb{R}^n$:

Theorem

Let Ω be a proper open subset in $\mathbb{R}^n$. Then there exists a countable collection of $\mathcal{Q}$ of essentially disjoint cubes, such that

$$\Omega = \bigcup_{Q \in \mathcal{Q}} Q,$$

with

$$diam(Q) \leq dist(Q, \mathbb{R}^n \setminus \Omega) < 4diam(Q).$$

- The proof of Whitney's theorem is just one sentence: Indeed one just takes $\mathcal{Q}$ to be the collection of maximal dyadic cubes in $\mathbb{R}^n$ that satisfies $\mathrm{diam}(Q) \leq \mathrm{dist}(Q, \mathbb{R}^n \setminus \Omega)$.
- Given $f \in L^1(\mathbb{R}^n)$ and $\alpha > 0$, we apply Whitney's theorem to the open set $\{x \in \mathbb{R}^n \colon Mf(x) > \alpha\}$.

- This yields a countable collection of $\mathcal{Q}$ of essentially disjoint cubes, for which

$$\sum_j |Q_j| = |\Omega| \leq \frac{C_n}{\alpha} \|f\|_{L^1},$$

  and for which

$$\fint_{Q_j} |f| \leq C_n \alpha \quad \text{for every } j$$

  since one can bound $\fint_{Q_j} |f|$ by $C_n \fint_{\tilde{Q}_j} |f|$, where $\tilde{Q}_j$ is a cube centered at some point in $\mathbb{R}^n \setminus \Omega$, of side length $\leq 5\text{diam}(Q)$, that contains Q_j (such $\tilde{Q}_j$ exists because of the distance comparison property in the Whitney decomposition theorem).
- We can then construct g and b from $\mathcal{Q}$ as before, and obtain a Calderón-Zygmund decomposition of f at height α.

Mapping properties of singular integral operators

Theorem

Let $K \in \mathcal{S}'(\mathbb{R}^n)$. Suppose $\widehat{K} \in L^\infty(\mathbb{R}^n)$ with

$$\|\widehat{K}\|_{L^\infty(\mathbb{R}^n)} \leq C,$$

and that K agrees with a C^1 function K_0 away from the origin, with

$$|\nabla K_0(x)| \leq C|x|^{-(n+1)} \quad \text{for all } x \neq 0.$$

Let $T \colon \mathcal{S}(R^n) \to \mathcal{S}'(\mathbb{R}^n)$ be the convolution operator defined by $Tf := f * K$. Then for all $f \in \mathcal{S}(\mathbb{R}^n)$, we have

$$|\{x \in \mathbb{R}^n \colon |Tf(x)| > \alpha\}| \leq \frac{C_n}{\alpha}\|f\|_{L^1(\mathbb{R}^n)} \quad \text{for all } \alpha > 0, \text{ and}$$

$$\|Tf\|_{L^p(\mathbb{R}^n)} \leq C_{n,p}\|f\|_{L^p(\mathbb{R}^n)} \quad \text{for all } 1 < p < \infty.$$

Hence T can be extended as an operator of weak-type $(1,1)$, and of strong-type (p,p) for all $1 < p < \infty$.

- The Hilbert transform and the Riesz transforms clearly fall under the scope of the previous theorem.
- Indeed $\widehat{K}$ is a homogeneous function of degree 0 (hence L^∞), and $K_0(x)$ is just a multiple of $\frac{1}{x}$ and $\frac{x_j}{|x|^{n+1}}$ respectively.
- Such K_0 are smooth on $\mathbb{S}^{n-1}$ and homogeneous of degree $-n$ (so that $\nabla K_0(x)$ satisfies the desired bound).
- Hence the theorem shows that the Hilbert transform and the Riesz transforms extend as an operator of weak-type $(1,1)$, and as a bounded linear operators on L^p, for $1 < p < \infty$.
- We note that the hypothesis on K_0 implies

$$\sup_{y \in \mathbb{R}^n} \int_{|x| \geq 2|y|} |K_0(x-y) - K_0(x)| dx \leq C;$$

  indeed this is all we will use in the proof.
- We now turn to the proof of this theorem.

- The proof of the theorem consists of four parts:
- First we prove the case $p = 2$;
  then we prove that T is weak-type $(1, 1)$;
  then we prove that T is bounded on L^p when $1 < p < 2$;
  finally we prove that T is bounded on L^p when $2 < p < \infty$.
- The case $p = 2$ follows from Plancherel easily, since we assumed $\widehat{K} \in L^\infty$:

$$\|Tf\|_{L^2} = \|\widehat{f}\widehat{K}\|_{L^2} \leq \|\widehat{K}\|_{L^\infty}\|\widehat{f}\|_{L^2} \leq C\|f\|_{L^2}.$$

- Next, to prove T is weak-type $(1, 1)$, let $f \in L^1$, $\alpha > 0$.
- Perform a Calderón-Zygmund decomposition at height α:

$$f = g + b = g + \sum_j b_j$$

  with b_j supported on Q_j for each j.

- We estimate

$$
\begin{aligned}
&|\{x \in \mathbb{R}^n \colon |Tf(x)| > \alpha\}| \\
\leq &|\{x \in \mathbb{R}^n \colon |Tg(x)| > \alpha/2\}| + |\{x \in \mathbb{R}^n \colon |Tb(x)| > \alpha/2\}|.
\end{aligned}
$$

- We have both $g \in L^1$ and $g \in L^\infty$, so $g \in L^2$ with

$$\|g\|_{L^2}^2 \leq C\alpha \|f\|_{L^1}.$$

This gives

$$|\{x \in \mathbb{R}^n \colon |Tg(x)| > \alpha/2\}| \leq \frac{4}{\alpha^2} \|Tg\|_{L^2}^2 \leq \frac{C}{\alpha} \|f\|_{L^1}.$$

- Also, if Q_j^* is the cube with same center as Q_j but $2\sqrt{n}$ times the side length, then $\Omega^* := \bigcup_j Q_j^*$ satisfies

$$|\Omega^*| \leq \sum_j |Q_j^*| \leq (2\sqrt{n})^n \sum_j |Q_j| \leq \frac{C_n}{\alpha} \|f\|_{L^1}.$$

- Thus we are left to show that

$$|\{x \in \mathbb{R}^n \setminus \Omega^* : |Tb(x)| > \alpha/2\}| \leq \frac{C}{\alpha}\|f\|_{L^1}.$$

- We do so by showing that

$$\|Tb_j\|_{L^1(\mathbb{R}^n \setminus Q_j^*)} \leq C\|b_j\|_{L^1} \quad \text{for all } j,$$

so that

$$\|Tb\|_{L^1(\mathbb{R}^n \setminus \Omega^*)} \leq C \sum_j \|Tb_j\|_{L^1(\mathbb{R}^n \setminus Q_j^*)} \leq C \sum_j \|b_j\|_{L^1} \leq C\|f\|_{L^1},$$

and the desired inequality follows by Chebyshev's inequality.

- To prove that

$$\|Tb_j\|_{L^1(\mathbb{R}^n \setminus Q_j^*)} \le C\|b_j\|_{L^1} \quad \text{for all } j,$$

recall $\int_{Q_j} b_j = 0$. Thus for $x \notin Q_j^*$, we have

$$\begin{aligned} Tb_j(x) &= \int_{y \in Q_j} K_0(x-y) b_j(y) dy \\ &= \int_{y \in Q_j} \left[K_0((x-y_j)-(y-y_j)) - K_0(x-y_j)\right] b_j(y) dy. \end{aligned}$$

where y_j is the center of Q_j. Note that $|x-y_j| \ge 2|y-y_j|$ if $x \notin Q_j^*$ and $y \in Q_j$. Also recall

$$\int_{|x| \ge 2|y|} |K_0(x-y) - K_0(x)| dx \le C$$

for all $y \in \mathbb{R}^n$.

- This shows

$$
\begin{aligned}
&\int_{\mathbb{R}^n \setminus Q_j^*} |Tb_j(x)| dx \\
\leq &\int_{y \in Q_j} \int_{x \in \mathbb{R}^n \setminus Q_j^*} |K_0((x - y_j) - (y - y_j)) - K_0(x - y_j)||b_j(y)| dx dy \\
\leq &\int_{y \in Q_j} C |b_j(y)| dy \\
\leq &C \|b_j\|_{L^1}
\end{aligned}
$$

  as desired, and finishes the proof that T is of weak-type $(1, 1)$.
- To prove that T is strong-type (p, p) for $1 < p < 2$, we interpolate between $p = 1$ and $p = 2$.

- More precisely, let $f \in L^p(\mathbb{R}^n)$, $1 < p < 2$. For $\alpha > 0$, write

$$f = f\chi_{|f|\leq\alpha} + f\chi_{|f|>\alpha} = f_\alpha + f^\alpha,$$

so that $f_\alpha \in L^2$, $f^\alpha \in L^1$.

- We have

$$\begin{aligned}\|Tf\|_{L^p}^p &= \int_0^\infty p\alpha^{p-1}|\{x \in \mathbb{R}^n \colon |Tf(x)| > \alpha\}|d\alpha \\ &\leq \int_0^\infty p\alpha^{p-1}|\{x \in \mathbb{R}^n \colon |Tf_\alpha(x)| > \alpha/2\}|d\alpha \\ &\quad + \int_0^\infty p\alpha^{p-1}|\{x \in \mathbb{R}^n \colon |Tf^\alpha(x)| > \alpha/2\}|d\alpha\end{aligned}$$

- But

$$|\{x \in \mathbb{R}^n \colon |Tf_\alpha(x)| > \alpha/2\}| \le \frac{4}{\alpha^2}\|Tf_\alpha\|_{L^2}^2 \le \frac{C}{\alpha^2}\|f_\alpha\|_{L^2}^2,$$

so

$$\begin{aligned}
&\int_0^\infty p\alpha^{p-1}|\{x \in \mathbb{R}^n \colon |Tf_\alpha(x)| > \alpha/2\}|d\alpha \\
\le& C\int_0^\infty p\alpha^{p-1}\alpha^{-2}\|f_\alpha\|_{L^2}^2 d\alpha \\
\le& C\int_{\mathbb{R}^n} |f(x)|^2 \int_{|f(x)|}^\infty p\alpha^{p-3} d\alpha \\
=& C_p\|f\|_{L^p}^p.
\end{aligned}$$

(We used $p < 2$ in the last line.)

- Similarly

$$|\{x \in \mathbb{R}^n \colon |Tf^\alpha(x)| > \alpha/2\}| \leq \frac{C_n}{\alpha} \|f^\alpha\|_{L^1},$$

so

$$\begin{aligned}
&\int_0^\infty p\alpha^{p-1} |\{x \in \mathbb{R}^n \colon |Tf^\alpha(x)| > \alpha/2\}| d\alpha \\
\leq & C_n \int_0^\infty p\alpha^{p-1}\alpha^{-1} \|f^\alpha\|_{L^1} d\alpha \\
\leq & C_n \int_{\mathbb{R}^n} |f(x)| \int_0^{|f(x)|} p\alpha^{p-2} d\alpha \\
= & C_{n,p} \|f\|_{L^p}^p.
\end{aligned}$$

(We used $p > 1$ in the last line.)

- The above shows

$$\|Tf\|_{L^p} \leq C_{n,p}\|f\|_{L^p} \quad \text{whenever } 1 < p < 2,$$

  i.e. T is strong-type (p,p) for $1 < p < 2$.
- Finally we need to show that T is strong-type (p,p) for $2 < p < \infty$.
- This follows by duality.
- Indeed the adjoint T^* of T also satisfies the same conditions as T.
- Hence T^* is of strong type (p,p) for $1 < p < 2$.
- It follows that T is of strong type (p,p) for $2 < p < \infty$.
- This finishes the proof of the theorem.
- We formulate a variable coefficient extension of the theorem in the next slide. It allows for operators that are not convolutions (that do not commute with translations).

Theorem

Let T be a bounded linear operator on $L^2(\mathbb{R}^n)$. Suppose there exists a locally L^∞ function K_0 on $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\}$, such that

$$Tf(x) = \int_{\mathbb{R}^n} f(y) K_0(x,y) dy$$

for every $f \in L^1(\mathbb{R}^n)$ with compact support, and a.e. $x \notin supp(f)$. Suppose in addition that

$$\sup_{(y,y_0) \in \mathbb{R}^n \times \mathbb{R}^n} \int_{|x-y_0| \geq 2|y-y_0|} |K_0(x,y) - K_0(x,y_0)| dx \leq C.$$

Then T extends as a linear operator of weak-type $(1,1)$, and of strong-type (p,p) for all $1 < p < \infty$.

- The proof is almost the same as before, which we omit.
- In Lecture 7 we will give some general conditions under which such T would be bounded on $L^2(\mathbb{R}^n)$.

Hörmander-Mikhlin multipliers

- We return to operators that commute with translations, and consider multiplier operators.
- Recall that if m is a bounded function on $\mathbb{R}^n$, then the operator
$$f \mapsto T_m f := \mathcal{F}^{-1}(m\widehat{f})$$
is bounded and linear on $L^2(\mathbb{R}^n)$.
- Such operators are called multiplier operators, and can be written as
$$T_m f = f * K$$
whenever $f \in \mathcal{S}(\mathbb{R}^n)$, where $K := \mathcal{F}^{-1}m$ is the inverse Fourier transform of the tempered distribution m.
- We seek conditions on m so that T_m extends as a bounded linear operator on $L^p(\mathbb{R}^n)$, for $1 < p < \infty$.

Theorem (Hörmander-Mikhlin)

Suppose m is a C^∞ function on $\mathbb{R}^n \setminus \{0\}$, and that

$$|\partial_\xi^\alpha m(\xi)| \lesssim_\alpha |\xi|^{-|\alpha|} \quad \text{for all } \xi \neq 0$$

and all multiindices α. Then T_m extends as a linear operator of weak-type $(1,1)$, and of strong-type (p,p) for all $1 < p < \infty$.

- This applies, for instance, when m is homogeneous of degree 0 and smooth on the unit sphere $\mathbb{S}^{n-1}$.
- In particular, this shows again that the Hilbert transform and the Riesz transforms are of weak-type $(1,1)$, and of strong-type (p,p) for all $1 < p < \infty$.
- Theorem also applies to imaginary powers of the Laplacian: $(-\Delta)^{it}$ is the multiplier operator with multiplier $(4\pi^2|\xi|^2)^{it}$, where the principal branch of the logarithm is taken ($t \in \mathbb{R}$).
- Various refinements of this theorem are given in Homework 4.

- The previous theorem deals with multipliers that are singular only at one point (the origin). Examples of multipliers with a bigger set of singularities will be considered in Lecture 11.
- The previous theorem also fails to deal with multipliers $m(\xi)$ that oscillates rapidly as $|\xi| \to \infty$. Examples of such rapidly oscillating multipliers include $e^{i|\xi|^a}$ when $a > 0$; such will be considered in Homework 9.
- The proof of the Hörmander-Mikhlin multiplier theorem consists of estimation of the convolution kernel $K := \mathcal{F}^{-1} m$.
- In particular, we show that K agrees with a C^∞ function K_0 away from the origin, and that

$$|\partial_x^\alpha K_0(x)| \lesssim_\alpha \frac{1}{|x|^{n+|\alpha|}} \quad \text{for all } x \neq 0$$

and all multiindices α, so that our previous theorem applies.

- Let $\psi(\xi)$ be a smooth function with compact support on the unit ball $B(0,2)$, with $\psi(\xi) \equiv 1$ on $B(0,1)$.
- Let $\varphi(\xi) = \psi(\xi) - \psi(2\xi)$ so that ψ is supported on the annulus $\{1/2 \leq |\xi| \leq 2\}$, and

$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1 \quad \text{for every } \xi \neq 0.$$

- Then

$$\sum_{|j| \leq J} \varphi(2^{-j}\xi) m(\xi) \to m(\xi)$$

  in the topology of $\mathcal{S}'(\mathbb{R}^n)$, as $J \to +\infty$.

- For $j \in \mathbb{Z}$, let

$$K^{(j)} = \mathcal{F}^{-1}[\varphi(2^{-j}\xi)m(\xi)] \in \mathcal{S}(\mathbb{R}^n)$$

so that $K := \mathcal{F}^{-1}m$ is the limit of $\sum_{|j|\leq J} K^{(j)}(x)$ in the topology of $\mathcal{S}'(\mathbb{R}^n)$ as $J \to +\infty$.

- But it is easy to see that

$$|\partial_x^\alpha K^{(j)}(x)| \lesssim_{\alpha,N} 2^{j(n+|\alpha|)} \min\{1, 2^{-jN}|x|^{-N}\}$$

for any multiindex α and any positive integer N.

- Thus there exists a C^∞ function $K_0(x)$ on $\mathbb{R}^n \setminus \{0\}$, so that

$$\sum_{|j|\leq J} K^{(j)}(x) \text{ converges uniformly to } K_0(x)$$

on any compact subsets of $\mathbb{R}^n \setminus \{0\}$ as $J \to \infty$.

- This shows $K := \mathcal{F}^{-1} m$ agrees with this C^∞ function K_0 away from the origin.
- Furthermore, the above estimates for $\partial_x^\alpha K^{(j)}(x)$ also readily implies that

$$|\partial_x^\alpha K_0(x)| \lesssim_\alpha \frac{1}{|x|^{n+|\alpha|}} \text{for all } x \neq 0$$

  and all multiindices α.
- Hence our previous theorem applies, and this concludes the proof of the Hörmander-Mikhlin multiplier theorem.

A vector-valued version of the main theorem

- We turn to a version of the singular integral theorem for vector-valued operators.
- Let B_1, B_2 be Banach spaces.
- Let $\mathrm{End}(B_1, B_2)$ be the space of continuous endomorphisms from B_1 to B_2.
- Let $L^p(\mathbb{R}^n, B_j)$ be the space of L^p mappings from $\mathbb{R}^n$ into B_j.
- Write $(\mathbb{R}^n \times \mathbb{R}^n)^*$ for the set $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \colon x \neq y\}$.
- Then we have the following theorem.

Theorem

Let T be a bounded linear operator from $L^q(\mathbb{R}^n, B_1)$ to $L^q(\mathbb{R}^n, B_2)$ for some $q \in (1, \infty]$. Suppose there exists a function

$$K_0(x, y) \in L^\infty_{loc}((\mathbb{R}^n \times \mathbb{R}^n)^*, End(B_1, B_2)),$$

such that

$$Tf(x) = \int_{\mathbb{R}^n} K_0(x, y) f(y) dy$$

for every $f \in L^1(\mathbb{R}^n, B_1)$ with compact support, and for a.e. $x \notin supp(f)$. Suppose in addition that

$$\sup_{(y, y_0) \in \mathbb{R}^n \times \mathbb{R}^n} \int_{|x - y_0| \geq 2|y - y_0|} \|K_0(x, y) - K_0(x, y_0)\|_{End(B_1, B_2)} dx \leq C.$$

Then T extends as a continuous linear operator from $L^1(\mathbb{R}^n, B_1)$ to $L^{1,\infty}(\mathbb{R}^n, B_2)$, and a continuous linear operator from $L^p(\mathbb{R}^n, B_1)$ to $L^p(\mathbb{R}^n, B_2)$ for all $1 < p < q$.

- The proof is almost the same as before, which we omit.
- Note that we do not claim mapping properties on L^p for $p > q$, because duality no longer works when say $q < 2$.
- We use this vector-valued version with $B_1 = \mathbb{C}$ and $B_2 = \ell^2(\mathbb{Z}, \mathbb{C})$ (or the other way round) to derive a Littlewood-Paley inequality.
- First we introduce the Littlewood-Paley projections in the next slide.

Littlewood-Paley decompositions

- As before, let $\psi(\xi)$ be a smooth function with compact support on the unit ball $B(0,2)$, with $\psi(\xi) \equiv 1$ on $B(0,1)$.
- Let $\varphi(\xi) = \psi(\xi) - \psi(2\xi)$ so that ψ is supported on the annulus $\{1/2 \leq |\xi| \leq 2\}$, and

$$\psi(\xi) + \sum_{j=1}^{\infty} \varphi(2^{-j}\xi) = 1 \quad \text{for every } \xi \in \mathbb{R}^n.$$

- For $f \in \mathcal{S}'(\mathbb{R}^n)$, let

$$P_0 f = \mathcal{F}^{-1}[\psi(\xi)\widehat{f}(\xi)], \qquad \text{and}$$

$$P_j f = \mathcal{F}^{-1}[\varphi(2^{-j}\xi)\widehat{f}(\xi)] \quad \text{for } j \geq 1.$$

- We think of $P_j f$ as the localization of f to frequency $\simeq 2^j$ if $j \geq 1$, and to frequency $\lesssim 1$ if $j = 0$.

- By Plancherel, it is easy to see that if $f \in L^2(\mathbb{R}^n)$, then $\sum_{j=0}^N P_j f$ converges to f in $L^2(\mathbb{R}^n)$ as $N \to \infty$.
- In addition, if $f \in \mathcal{S}(\mathbb{R}^n)$, then $\sum_{j=0}^N P_j f$ converges to f in the topology of $\mathcal{S}(\mathbb{R}^n)$ as $N \to \infty$.
- As a result, if $f \in \mathcal{S}'(\mathbb{R}^n)$, then $\sum_{j=0}^N P_j f$ converges to f in the topology of $\mathcal{S}'(\mathbb{R}^n)$ as $N \to \infty$.
- This applies, in particular, for every $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$.
- Note also that if $f \in \mathcal{S}'(\mathbb{R}^n)$, then

$$\sum_{j=0}^{N} P_j f = f * k_{2^{-N}}$$

  where $k_\varepsilon(x) := \varepsilon^{-n} k(\varepsilon^{-1} x)$, and $k := \mathcal{F}^{-1}\psi$.
- Thus $\sum_{j=0}^N P_j f(x)$ converges pointwisely to $f(x)$ for a.e. $x \in \mathbb{R}^n$, whenever $f \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq \infty$.
- If in addition $1 \leq p < \infty$, then Question 3 from Homework 3 then shows that for every $f \in L^p(\mathbb{R}^n)$, $\sum_{j=0}^N P_j f$ converges to f in $L^p(\mathbb{R}^n)$ as $N \to \infty$.

- Thus for $1 \leq p < \infty$, one way of estimating the L^p norm of $f \in L^p(\mathbb{R}^n)$ is to estimate

$$\sup_{N \geq 1} \left\| \sum_{j=0}^{N} P_j f \right\|_{L^p(\mathbb{R}^n)}.$$

- Via the triangle inequality, we can control the above expression if we can bound

$$\left\| \sum_{j=0}^{N} |P_j f| \right\|_{L^p(\mathbb{R}^n)}$$

uniformly in N, which can be done if we can bound

$$N^{1/2} \left\| \left(\sum_{j=0}^{N} |P_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)}$$

uniformly in N.

- It turns out that one can do better, when $1 < p < \infty$; intuitively this is because there is certain orthogonality between the different $P_j f$'s. (In particular, the following Theorem is easy to prove if $p = 2$, by Plancherel.)

Theorem (Littlewood-Paley)

Suppose $1 < p < \infty$.

(a) *For every* $f \in L^p(\mathbb{R}^n)$, *we have*

$$\|f\|_{L^p(\mathbb{R}^n)} \simeq \left\| \left(\sum_{j=0}^{\infty} |P_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} .$$

(b) *Furthermore, if* $f \in \mathcal{S}'(\mathbb{R}^n)$ *and the right hand side above is finite, then* $f \in L^p(\mathbb{R}^n)$ *(hence the above comparison holds).*

- Let's first prove that

$$\left\| \left(\sum_{j=0}^{\infty} |P_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$$

  when $f \in L^p(\mathbb{R}^n)$ and $1 < p < \infty$.
  (This is half of part (a) of the Theorem.)
- Note that the term corresponding to $j = 0$ can be easily estimated: indeed

$$\|P_0 f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}$$

  by Young's convolution inequality.
- Thus we may replace $\sum_{j=0}^{\infty}$ by $\sum_{j=1}^{\infty}$.
- The crux of the matter is captured in the following theorem:

Theorem'

Let $\Phi \in \mathcal{S}(\mathbb{R}^n)$ *with* $\int_{\mathbb{R}^n} \Phi = 0$. *For* $j \in \mathbb{Z}$ *and* $f \in \mathcal{S}'(\mathbb{R}^n)$, *let*

$$\Delta_j f(x) = f * \Phi_j(x) \quad \text{where } \Phi_j(x) = 2^{jn}\Phi(2^j x).$$

Let $1 < p < \infty$. *Then for all* $f \in L^p(\mathbb{R}^n)$, $\|\Delta_j f\|_{\ell^2(\mathbb{Z})} \in L^p(\mathbb{R}^n)$, *and*

$$\Big\| \|\Delta_j f\|_{\ell^2(\mathbb{Z})} \Big\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|f\|_{L^p(\mathbb{R}^n)}.$$

- By monotone convergence, density and Fatou's lemma, we may assume $f \in \mathcal{S}(\mathbb{R}^n)$.
- We will treat the cases $1 < p \leq 2$ and $2 \leq p < \infty$ separately.
- For $1 < p \leq 2$, let $B_1 = \mathbb{C}$, $B_2 = \ell^2(\mathbb{Z}, \mathbb{C})$ = the space of all complex-valued ℓ^2 sequences $(a_j)_{j \in \mathbb{Z}}$.
- We apply the vector-valued singular integral theorem.

- Let $K(x) = \{\Phi_j(x)\}_{j\in\mathbb{Z}} \in \mathrm{End}(B_1, B_2)$ and $Tf(x) = f * K(x)$ so that $Tf(x) = \{\Delta_j f(x)\}_{j\in\mathbb{Z}}$.
- One checks, via Plancherel, that T defines a bounded linear map from $L^2(\mathbb{R}^n, B_1)$ to $L^2(\mathbb{R}^n, B_2)$; also

$$\|\partial_x K(x)\|_{\mathrm{End}(B_1,B_2)} = \left(\sum_{j\in\mathbb{Z}} |\partial_x \Phi_j(x)|^2 \right)^{1/2}$$

is bounded by $C|x|^{-(n+1)}$ whenever $x \neq 0$.
- Thus the vector-valued singular integral theorem shows that

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\Delta_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|f\|_{L^p(\mathbb{R}^n)}$$

whenever $1 < p \leq 2$ and $f \in \mathcal{S}(\mathbb{R}^n)$, as desired.

- For $2 \le p < \infty$, we observe that by duality, we just need to prove that

$$\left\| \sum_{j \in \mathbb{Z}} \Delta_j g_j \right\|_{L^{p'}(\mathbb{R}^n)} \lesssim_{n,p} \left\| \|g_j\|_{\ell^2(\mathbb{Z})} \right\|_{L^{p'}(\mathbb{R}^n)}$$

for all sequences of Schwartz functions $\{g_j\}_{j \in \mathbb{Z}}$, where only finitely many g_j's are non-zero.
- This is because then for every $f \in L^p(\mathbb{R}^n)$ and all such $\{g_j\}$'s, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \sum_{j \in \mathbb{Z}} \Delta_j f(x) g_j(x) dx &= \int_{\mathbb{R}^n} f(x) \sum_{j \in \mathbb{Z}} \Delta_j g_j(x) dx \\ &\lesssim \|f\|_{L^p(\mathbb{R}^n)} \left\| \sum_{j \in \mathbb{Z}} \Delta_j g_j \right\|_{L^{p'}(\mathbb{R}^n)} \\ &\lesssim_{n,p} \|f\|_{L^p(\mathbb{R}^n)} \left\| \|g_j\|_{\ell^2(\mathbb{Z})} \right\|_{L^{p'}(\mathbb{R}^n)}.\end{aligned}$$

The density of such $\{g_j\}$'s in $L^{p'}(\ell^2)$ gives the desired conclusion.

$$\left\| \sum_{j\in\mathbb{Z}} \Delta_j g_j \right\|_{L^{p'}(\mathbb{R}^n)} \lesssim_{n,p} \left\| \|g_j\|_{\ell^2(\mathbb{Z})} \right\|_{L^{p'}(\mathbb{R}^n)}$$

- To prove this, this time we let $B_1 = \ell^2(\mathbb{Z}, \mathbb{C})$, $B_2 = \mathbb{C}$, and apply the vector-valued singular integral theorem.
- Indeed, let $K(x) = \sum_{j\in\mathbb{Z}} \Phi_j(x) e_j^* \in \text{End}(B_1, B_2)$ where $\{e_j\}$ is the coordinate basis of $B_1 = \ell^2$, and $\{e_j^*\}$ is the dual basis. Let

$$\tilde{T}g(x) = \int_{y\in\mathbb{R}^n} K(y) g(x-y) dy$$

if $g = \{g_j\}$, so that

$$\tilde{T}g(x) = \sum_{j\in\mathbb{Z}} \Delta_j g_j.$$

- One checks, via Plancherel, that $\tilde{T}$ defines a bounded linear map from $L^2(\mathbb{R}^n, B_1)$ to $L^2(\mathbb{R}^n, B_2)$; also

$$\|\partial_x K(x)\|_{\mathrm{End}(B_1,B_2)} = \left(\sum_{j\in\mathbb{Z}} |\partial_x \Phi_j(x)|^2 \right)^{1/2}$$

  is bounded by $C|x|^{-(n+1)}$ whenever $x \neq 0$.
- Thus the vector-valued singular integral theorem says that

$$\left\| \sum_{j\in\mathbb{Z}} \Delta_j g_j \right\|_{L^{p'}(\mathbb{R}^n)} \lesssim_{n,p} \left\| \|g_j\|_{\ell^2(\mathbb{Z})} \right\|_{L^{p'}(\mathbb{R}^n)}$$

  for $1 < p' \leq 2$, i.e. for $2 \leq p < \infty$, as desired.

- Theorem' can also be proved using Klintchine's inequality, without using vector-valued singular integrals.
- For $j \geq 1$, define the j-th Rademacher function r_j by

$$r_j(t) = \begin{cases} +1 & \text{if } t \in [k2^{-j}, (k+1)2^{-j}) \text{ for some odd integer } k \\ -1 & \text{if } t \in [k2^{-j}, (k+1)2^{-j}) \text{ for some even integer } k \end{cases}$$

Theorem (Klintchine)

For every $p \in (0, \infty)$, there exist constants A_p and B_p that depend only on p (but not on N), such that for every sequence of complex numbers $\{a_j\}_{1 \leq j \leq N}$, we have

$$A_p \left(\sum_{j=1}^{N} |a_j|^2 \right)^{p/2} \leq \int_0^1 \left| \sum_{j=1}^{N} a_j r_j(t) \right|^p dt \leq B_p \left(\sum_{j=1}^{N} |a_j|^2 \right)^{p/2}.$$

- A more general version is given in Homework 4.

- We are now ready to give a second proof of Theorem'.
- Let $\{\varepsilon_j(t)\}_{j\in\mathbb{Z}}$ be an enumeration of $\{r_j(t)\}_{j\geq 1}$.

Proposition

Let $\Phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \Phi = 0$. For $j \in \mathbb{Z}$ and $f \in \mathcal{S}(\mathbb{R}^n)$, let

$$\Delta_j f(x) = f * \Phi_j(x) \quad \text{where } \Phi_j(x) = 2^{jn}\Phi(2^j x).$$

Then for every $1 < p < \infty$ and $f \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\sup_{N\in\mathbb{N}} \sup_{t\in[0,1]} \left\| \sum_{|j|\leq N} \varepsilon_j(t)\Delta_j f \right\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|f\|_{L^p(\mathbb{R}^n)}.$$

- If the proposition were true, then applying Klintchine, we have

$$\sup_{N\in\mathbb{N}} \left\| \left(\sum_{|j|\leq N} |\Delta_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|f\|_{L^p(\mathbb{R}^n)}$$

for $1 < p < \infty$ and $f \in \mathcal{S}(\mathbb{R}^n)$.

- The same holds for $f \in L^p(\mathbb{R}^n)$ by density, which gives our desired conclusion.

- To prove the proposition, for every $t \in [0,1]$, and every $N \in \mathbb{N}$, let

$$K_{N,t}(x) = \sum_{|j| \leq N} \varepsilon_j(t) \Phi_j(x)$$

so that

$$\sum_{|j| \leq N} \varepsilon_j(t) \Delta_j f = f * K_{N,t}.$$

- One checks that $K_{N,t}$ is a Calderon-Zygmund kernel uniformly in N and t.
- The (scalar-valued) singular integral theorem then gives the claim of the proposition.

- To recap, we proved half of part (a) of the Littlewood-Paley theorem. It says that for every $1 < p < \infty$, $f \in L^p(\mathbb{R}^n)$, we have
$$\left\| \left(\sum_{j=0}^{\infty} |P_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|f\|_{L^p(\mathbb{R}^n)}.$$
- We remark that the smoothness of the multipliers for P_j is important here; for instance, if we defined P_0 instead by
$$P_0 f = \mathcal{F}^{-1}(\chi_{B(0,1)}(\xi)\widehat{f}(\xi)),$$
where $\chi_{B(0,1)}$ is the characteristic function of the unit ball, then P_0 is not bounded on $L^p(\mathbb{R}^n)$ whenever $n \geq 2$ and $p \neq 2$.
- The latter is the famous ball multiplier theorem of Fefferman, to which we will return in Lecture 11.

- We still need to prove the other half of part (a), and also part (b), of the Littlewood-Paley theorem.
- For that we use duality.
- For $f \in \mathcal{S}'(\mathbb{R}^n)$, let $P_j f$ be the Littlewood-Paley projection of f defined earlier, for $j \geq 0$.
- We need to prove the following: If $f \in \mathcal{S}'(\mathbb{R}^n)$ and

$$\left\| \left(\sum_{j=0}^{\infty} |P_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} < \infty$$

for some $1 < p < \infty$, then f can be identified with an L^p function on $\mathbb{R}^n$, and that $\|f\|_{L^p(\mathbb{R}^n)}$ is controlled by the above quantity.

- Indeed, it suffices to show that

$$\langle f, g\rangle \lesssim_{n,p} \left\| \left(\sum_{j=0}^{\infty} |P_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \|g\|_{L^{p'}(\mathbb{R}^n)}$$

  for all $g \in \mathcal{S}(\mathbb{R}^n)$.
- To do so, let $\tilde{\psi}$ be a smooth function with compact support on the unit ball $B(0,4)$, with $\tilde{\psi} \equiv 1$ on the support of ψ.
- Let $\tilde{\varphi}$ be a smooth function with compact support on the annulus $\{1/4 \leq |\xi| \leq 4\}$, with $\tilde{\varphi} \equiv 1$ on the support of φ.
- For $g \in \mathcal{S}(\mathbb{R}^n)$, let

$$\tilde{P}_0 g = \mathcal{F}^{-1}[\tilde{\psi}(\xi)\widehat{g}(\xi)], \quad \text{and}$$

$$\tilde{P}_j g = \mathcal{F}^{-1}[\tilde{\varphi}(2^{-j}\xi)\widehat{g}(\xi)] \quad \text{for } j \geq 1.$$

- Note that if $j \geq 1$, then $\tilde{P}_j g = g * \tilde{\Phi}_{2^{-j}}$ for some Schwartz function $\tilde{\Phi}$ with $\int_{\mathbb{R}^n} \tilde{\Phi} = 0$, so the forward Littlewood-Paley inequality applies. Also $P_j = P_j \tilde{P}_j$.

- Hence for $g \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\begin{aligned}
\langle f, g\rangle &= \lim_{N\to\infty} \sum_{j=0}^{N} \langle f, P_j g\rangle = \lim_{N\to\infty} \sum_{j=0}^{N} \langle f, P_j \tilde{P}_j g\rangle \\
&= \lim_{N\to\infty} \sum_{j=0}^{N} \langle P_j f, \tilde{P}_j g\rangle \\
&\leq \left\| \left(\sum_{j=0}^{\infty} |P_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \left\| \left(\sum_{j=0}^{\infty} |\tilde{P}_j g|^2 \right)^{1/2} \right\|_{L^{p'}(\mathbb{R}^n)} \\
&\lesssim_{n,p} \left\| \left(\sum_{j=0}^{\infty} |P_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \|g\|_{L^{p'}(\mathbb{R}^n)}.
\end{aligned}$$

This concludes the proof of the other half of part (a), and also part (b), of the Littlewood-Paley theorem we stated earlier.

- To close this lecture, we remark that if $f \in \mathcal{S}'(\mathbb{R}^n)$ and
$$\left\| \left(\sum_{j \in \mathbb{Z}} |\Delta_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} < \infty$$
for some $1 < p < \infty$, it does NOT necessarily follow that f can be identified with an L^p function on $\mathbb{R}^n$. This is because Δ_j does not capture the Fourier transform of f at 0.
- Nonetheless, if $1 < p < \infty$, and if we already know that $f \in L^p(\mathbb{R}^n)$, then we do have
$$\|f\|_{L^p(\mathbb{R}^n)} \lesssim \left\| \left(\sum_{j \in \mathbb{Z}} |\Delta_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)}.$$
- This is because for $g \in \mathcal{S}(\mathbb{R}^n)$, we do have $\sum_{|j| \leq N} \Delta_j g \to g$ in $L^{p'}(\mathbb{R}^n)$ as $N \to \infty$, when $1 < p' < \infty$, so that we can run our previous argument. c.f. Homework 4 for the former fact.