Topics in Harmonic Analysis
Lecture 5: Hölder spaces, Sobolev spaces and BMO functions

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Introduction

- In the previous two lectures, we discussed Riesz potentials, singular integrals and Littlewood-Paley projections.
- Today we will use these ideas, to study various function spaces that are important in the study of harmonic analysis and partial differential equations.

Outline

- Hölder spaces: equivalent characterizations
- Sobolev spaces: equivalent characterizations
- Sobolev and Morrey embedding theorems
- Functions of bounded mean oscillations (BMO)
- Singular integrals map L^∞ into BMO
- Sobolev embeddings into BMO
- The John-Nirenberg inequality for BMO

Hölder spaces

- Let $\gamma \in (0,1)$. A function f on $\mathbb{R}^n$ is said to be Hölder continuous of order γ (written $f \in \Lambda^\gamma$), if

$$|f|_{\Lambda^\gamma} := \|f\|_{L^\infty} + \sup_{\substack{x,y\in\mathbb{R}^n \\ y\neq 0}} \frac{|f(x+y)-f(x)|}{|y|^\gamma} < +\infty.$$

- We will see shortly a characterization of Λ^γ by the Littlewood-Paley projections we introduced last time.
- As before, let $\psi(\xi)$ be a smooth function with compact support on the unit ball $B(0,2)$, with $\psi(\xi) \equiv 1$ on $B(0,1)$.
- Let $\varphi(\xi) = \psi(\xi) - \psi(2\xi)$ so that ψ is supported on the annulus $\{1/2 \leq |\xi| \leq 2\}$.
- For $f \in \mathcal{S}'(\mathbb{R}^n)$, let

$$P_0 f = \mathcal{F}^{-1}[\psi(\xi)\widehat{f}(\xi)], \quad \text{and}$$

$$P_j f = \mathcal{F}^{-1}[\varphi(2^{-j}\xi)\widehat{f}(\xi)] \quad \text{for } j \geq 1.$$

Theorem

Let $\gamma \in (0,1)$, and $f \in \mathcal{S}'(\mathbb{R}^n)$. Then $f \in \Lambda^\gamma$ (more precisely, the tempered distribution f is given by a function in Λ^γ) if and only if there exists a constant $C > 0$ such that

$$\|P_j f\|_{L^\infty(\mathbb{R}^n)} \leq C 2^{-j\gamma} \quad \textit{for all } j \geq 0,$$

and the smallest C for which this holds is comparable to $|f|_{\Lambda^\gamma}$.

- This allows one to extend the definition of Λ^γ to all $\gamma > 0$: For every $\gamma > 0$, we *define* the Hölder space $\Lambda^\gamma(\mathbb{R}^n)$ to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$, for which there exists C such that

  $$\|P_j f\|_{L^\infty(\mathbb{R}^n)} \leq C 2^{-j\gamma} \quad \text{for all } j \geq 0.$$

  The smallest C for which this holds is denoted $\|f\|_{\Lambda^\gamma}$. (Λ^γ is also sometimes called a Zygmund space; the next two theorems together show that $\|f\|_{\Lambda^\gamma}$ is well-defined up to a multiplicative constant, irrespective of the choice of P_j.)

- We sketch only the essence of the proof of the theorem. It relies on the following lemma (which can be proved by rescaling to the unit scale):

Lemma

For any multiindices β and any $j \geq 0$, we have

$$\|\partial^\beta P_j f\|_{L^\infty} \lesssim 2^{j|\beta|} \|P_j f\|_{L^\infty}$$

for all $f \in \mathcal{S}'(\mathbb{R}^n)$.

- So if $\|P_j f\|_{L^\infty} \leq C 2^{-j\gamma}$ for all $j \geq 0$, then

$$|f(x+y) - f(x)| \leq \sum_{j \geq 0} |P_j f(x+y) - P_j f(x)|,$$

which we estimate using

$$|P_j f(x+y) - P_j f(x)| \lesssim \begin{cases} 2\|P_j f\|_{L^\infty} \lesssim C 2^{-j\gamma} & \text{if } 2^j > |y|^{-1} \\ |y| \|\nabla P_j f\|_{L^\infty} \lesssim C|y| 2^{j(1-\gamma)} & \text{if } 2^j \leq |y|^{-1} \end{cases}.$$

- Conversely, for $j \geq 1$, we have

$$P_j f(x) = \int_{\mathbb{R}^n} f(x-y)\Phi_j(y)dy$$

if $\Phi_j(y) = 2^{jn}\Phi(2^j y)$ and $\Phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\widehat{\Phi} = \varphi$.
- Note that $\int \Phi = 0$. Thus if

$$|f(x+y) - f(x)| \leq C|y|^\gamma,$$

then for $j \geq 1$, we have

$$P_j f(x) = \int_{\mathbb{R}^n} [f(x-y) - f(x)]\Phi_j(y)dy,$$

from which it follows that

$$\|P_j f\|_{L^\infty} \leq C \int |y|^\gamma |\Phi_j(y)|dy \lesssim C2^{-j\gamma}.$$

- This finishes the sketch of the proof of the theorem.

- Let's call a function f on $\mathbb{R}^n$ Lipschitz, if f is bounded and

$$|f(x+y) - f(x)| \leq C|y|$$

for all $x, y \in \mathbb{R}^n$.

- One might guess that perhaps Λ^1 is the space of Lipschitz functions on $\mathbb{R}^n$, but it is not; the space of Lipschitz functions on $\mathbb{R}^n$ is only a proper subset of Λ^1 (e.g. $f(x) = \sum_{k=1}^{\infty} 2^{-k} e^{2\pi i 2^k x} \in \Lambda^1(\mathbb{R})$ but is not Lipschitz).
- Indeed, if $f \in \mathcal{S}'(\mathbb{R}^n)$, then $f \in \Lambda^1$ if and only if f is bounded and

$$|f(x+y) + f(x-y) - 2f(x)| \leq C|y|$$

for all $x, y \in \mathbb{R}^n$.

- More generally, we have the following theorem:

Theorem

Let $\gamma \in (0,2)$, and $f \in \mathcal{S}'(\mathbb{R}^n)$. Then $f \in \Lambda^\gamma$ if and only if

$$\|f\|_{L^\infty} + \sup_{\substack{x,y\in\mathbb{R}^n \\ y\neq 0}} \frac{|f(x+y)+f(x-y)-2f(x)|}{|y|^\gamma} < +\infty.$$

Also, the quantity above is comparable to $\|f\|_{\Lambda^\gamma}$.

- Combined with the following theorem, we can characterize Λ^γ using difference quotients only, for all $\gamma > 0$.

Theorem

Let $\gamma > 1$, $m \in \mathbb{N}$ be so that $\gamma - m \in (0,1]$, and $f \in \mathcal{S}'(\mathbb{R}^n)$. Then $f \in \Lambda^\gamma$ if and only if

$$\|f\|_{L^\infty} + \sum_{|\beta|=m} \|\partial^\beta f\|_{\Lambda^{\gamma-m}} < +\infty.$$

Also, the quantity above is comparable to $\|f\|_{\Lambda^\gamma}$.

- The proof of the first theorem on the previous slide is similar to the proof of the earlier theorem. The key is to note that

$$|P_j f(x+y) + P_j f(x-y) - 2P_j f(x)| \lesssim \|\nabla^2 P_j f\|_{L^\infty} |y|^2,$$

and that

$$P_j f(x) = \int_{\mathbb{R}^n} \frac{f(x+y) + f(x-y) - 2f(x)}{2} \Phi_j(y) dy$$

if Φ is even and $j \geq 1$. We omit the details.

- To prove the second theorem, if $f \in \Lambda^\gamma$, we just use our previous lemma to control $\|P_j(\partial^\beta f)\|_{L^\infty}$, which in turn yields a control of $\|\partial^\beta f\|_{\Lambda^{\gamma-m}}$ if $|\beta| = m$.
- Conversely, for $j \geq 1$, we have[1]

$$P_j f = \sum_{|\beta|=m} (-1)^m (-\Delta)^{-m} \partial^\beta P_j(\partial^\beta f),$$

so it remains to observe that

$$\|(-\Delta)^{-m} \partial^\beta P_j g\|_{L^\infty} \lesssim 2^{j(|\beta|-2m)} \|P_j g\|_{L^\infty} \quad \text{for all } g,$$

which can be proved the same way as our previous lemma. We omit the details.

[1] Technically $\partial^\beta P_j(\partial^\beta f)$ is only in $\mathcal{S}'(\mathbb{R}^n)$, and $(-\Delta)^{-m}$ is not defined for every element in $\mathcal{S}'(\mathbb{R}^n)$. Fortunately, the Fourier transform of $\partial^\beta P_j(\partial^\beta f)$ is supported away from the origin in the frequency space. So strictly speaking, instead of $(-\Delta)^{-m}$, we should write a multiplier operator whose multiplier agrees with that of $(-\Delta)^{-m}$ on the frequency support of $\partial^\beta P_j(\partial^\beta f)$. The observation that follows remains valid.

Sobolev spaces

- Let $1 \le p \le \infty$, and $f \in L^p(\mathbb{R}^n)$.
- Then $f \in \mathcal{S}'(\mathbb{R}^n)$, so it makes sense to consider its distributional derivatives $\partial^\beta f$ for any multiindices β.
- Suppose $k \in \mathbb{N}$, and $\partial^\beta f$ agrees with an L^p function on $\mathbb{R}^n$ for every multiindex β with $|\beta| \le k$.
- Then f is said to be in the Sobolev space $W^{k,p}(\mathbb{R}^n)$, and

$$\|f\|_{W^{k,p}} := \sum_{|\beta|\le k} \|\partial^\beta f\|_{L^p(\mathbb{R}^n)}.$$

- For $1 < p < \infty$, we can characterize Sobolev spaces by a close relative to the Riesz potentials, called Bessel potentials.
- For $\alpha \in \mathbb{R}$, the Bessel potential of order α is defined by

$$\mathcal{J}_\alpha f(x) := (I - \Delta)^{-\alpha/2} f(x)$$

  for $f \in \mathcal{S}'(\mathbb{R}^n)$; in other words, $\mathcal{J}_\alpha$ is the multiplier operator with multiplier $(1 + 4\pi^2|\xi|^2)^{-\alpha/2}$. Note that $\mathcal{J}_\alpha$ is a homeomorphism on the topological vector space $\mathcal{S}(\mathbb{R}^n)$.

Theorem

Let $k \in \mathbb{N}$, $1 < p < \infty$, and $f \in \mathcal{S}'(\mathbb{R}^n)$. Then $f \in W^{k,p}(\mathbb{R}^n)$, if and only if $(I - \Delta)^{k/2} f \in L^p$. Furthermore,

$$\|f\|_{W^{k,p}} \simeq \|(I - \Delta)^{k/2} f\|_{L^p}.$$

- This allows one to extend the definition of $W^{k,p}$, and define $W^{\alpha,p}$ for all $\alpha > 0$ and $1 < p < \infty$: for such α and p, we *define* the Sobolev space $W^{\alpha,p}(\mathbb{R}^n)$ by

$$W^{\alpha,p}(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) \colon (I - \Delta)^{\alpha/2} f \in L^p \right\},$$

and write $\|f\|_{W^{\alpha,p}}$ for any quantity $\simeq \|(I - \Delta)^{\alpha/2} f\|_{L^p}$.

- To prove the theorem, suppose $g := (I - \Delta)^{k/2} f \in L^p$. Then for any multiindex β, we have

$$\partial^\beta f = \partial^\beta \mathcal{J}_k g,$$

  and $\partial^\beta \mathcal{J}_k$ is bounded on L^p if $|\beta| \leq k$ and $1 < p < \infty$, by the Hörmander-Mikhlin multiplier theorem. So $\partial^\beta f \in L^p$ for all $|\beta| \leq k$, if $1 < p < \infty$.
- Conversely, let $\partial^\beta f \in L^p$ for $|\beta| \leq k$, and $1 < p < \infty$. Then

$$(I-P_0)(I-\Delta)^{k/2} f = \sum_{|\beta|=k} (-1)^k (I-\Delta)^{k/2}(I-P_0)(-\Delta)^{-k} \partial^\beta (\partial^\beta f),$$

  and $(I - \Delta)^{k/2}(-\Delta)^{-k}(I - P_0)\partial^\beta$ is bounded on L^p if $|\beta| = k$ and $1 < p < \infty$, by the theorem of Hörmander-Mikhlin (note that the cut-off $I - P_0$ vanishes near the origin).
- So $(I - P_0)(I - \Delta)^{k/2} f \in L^p$, and together with a trivial bound for $P_0(I - \Delta)^{k/2} f$, we see that $(I - \Delta)^{k/2} f \in L^p$.

- One can also characterize Sobolev spaces by Littlewood-Paley projections and *square functions* when $1 < p < \infty$:

Theorem

Let $\alpha > 0$, $1 < p < \infty$, and $f \in \mathcal{S}'(\mathbb{R}^n)$. Then $f \in W^{\alpha,p}(\mathbb{R}^n)$, if and only if

$$\left\| \left(\sum_{j=0}^{\infty} |2^{j\alpha} P_j f|^2 \right)^{1/2} \right\|_{L^p} < +\infty.$$

Furthermore, the quantity above is comparable to $\|f\|_{W^{\alpha,p}}$.

- Indeed, a vector-valued singular integral theorem shows that

$$\left\| \left(\sum_{j=0}^{\infty} |P_j (I - \Delta)^{\alpha/2} f|^2 \right)^{1/2} \right\|_{L^p} \simeq \left\| \left(\sum_{j=0}^{\infty} |2^{j\alpha} P_j f|^2 \right)^{1/2} \right\|_{L^p}$$

if $\alpha > 0$ and $1 < p < \infty$.

Sobolev and Morrey embedding theorems

- The Sobolev embedding theorem describes continuous embeddings of Sobolev spaces into appropriate L^q spaces.

Theorem (Sobolev embedding theorem)

(a) *If* $\alpha \in (0, n)$ *and* $1 < p < n/\alpha$, *then*

$$W^{\alpha,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n) \quad \textit{if} \quad \frac{1}{p^*} = \frac{1}{p} - \frac{\alpha}{n}.$$

(b) *If* $k \in (0, n)$ *is an integer, then*

$$W^{k,1}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n) \quad \textit{if} \quad \frac{1}{p^*} = 1 - \frac{k}{n}.$$

(c) *Also,*

$$W^{n,1}(\mathbb{R}^n) \hookrightarrow L^{\infty}(\mathbb{R}^n).$$

- To prove (a), it suffices to know that $\mathcal{J}_\alpha \colon L^p(\mathbb{R}^n) \to L^{p^*}(\mathbb{R}^n)$, if $\alpha \in (0, n)$, $1 < p < n/\alpha$ and $\frac{1}{p^*} = \frac{1}{p} - \frac{\alpha}{n}$. This follows by writing
$$\mathcal{J}_\alpha = P_0 \mathcal{J}_\alpha + (I - P_0)\mathcal{J}_\alpha (-\Delta)^{\alpha/2} \mathcal{I}_\alpha;$$
note that $P_0 \mathcal{J}_\alpha \colon L^p \to L^{p^*}$, $\mathcal{I}_\alpha \colon L^p \to L^{p^*}$, and $(I - P_0)\mathcal{J}_\alpha(-\Delta)^{\alpha/2}$ is bounded on L^{p^*} by the theorem of Hörmander-Mikhlin. (We put $I - P_0$ to make sure that $(I - P_0)\mathcal{J}_\alpha(-\Delta)^{\alpha/2}$ is well-defined from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$.)
- To prove (b), note that it suffices to prove the case $k = 1$ (when $n > 1$) and then use part (a). We use the following density theorem, which is of independent interest:

Theorem

$C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}$ for all $k \in \mathbb{N}$ and $1 \leq p < \infty$.

- Now for $f \in C_c^\infty(\mathbb{R}^n)$, we have

$$|f(x)|^{\frac{n}{n-1}} \lesssim \prod_{k=1}^{n} \|\partial_j f(x)\|_{L^1(dx_k)}^{\frac{1}{n-1}};$$

  this is a simple consequence of the fundamental theorem of calculus in one variable.
- The k-th factor on the right hand side is a function of $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n$, so it suffices to apply the following *Loomis-Whitney inequality* to conclude:

$$\int_{\mathbb{R}^n} \prod_{k=1}^{n} F_k(\pi_k(x))^{\frac{1}{n-1}} \, dx \leq \prod_{k=1}^{n} \|F_k\|_{L^1(\mathbb{R}^{n-1})}^{\frac{1}{n-1}}$$

  Here $F_1, \ldots, F_n$ are any n non-negative measurable functions on $\mathbb{R}^{n-1}$, and $\pi_k \colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the coordinate projection forgetting the k-th coordinate. (This Loomis-Whitney inequality is a simple consequence of Hölder's inequality.)

- The above shows that if $f \in C_c^\infty(\mathbb{R}^n)$, then

$$\|f\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq C_n \|\nabla f\|_{L^1(\mathbb{R}^n)}.$$

- This is called the Gagliardo-Nirenberg inequality on $\mathbb{R}^n$.
- The Gagliardo-Nirenberg inequality is known to be equivalent with the isoperimetric inequality in geometry, which says that for any bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary,

$$|\Omega|^{\frac{n-1}{n}} \leq D_n |\partial\Omega|.$$

- Indeed the best constants of the two inequalites are the same, which is achieved when Ω is a ball in $\mathbb{R}^n$.
- This best constant plays an important role in the study of many critical non-linear partial differential equations (see e.g. the Yamabe equation in conformal geometry).
- One finishes the proof of (b) by approximating a general $W^{1,1}(\mathbb{R}^n)$ function by C_c^∞ functions, and appealing to the Gagliardo-Nirenberg inequality.

- Finally to prove (c), note that for $f \in C_c^\infty(\mathbb{R}^n)$, then

$$f(x) = \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_n} [\partial_1 \dots \partial_n f](y) dy_n \dots dy_1$$

for all $x \in \mathbb{R}^n$. Hence

$$\|f\|_{L^\infty} \leq \|\partial_1 \dots \partial_n f\|_{L^1}.$$

The desired conclusion follows by approximation of $W^{n,1}$ functions by C_c^∞ functions.
(Indeed this also shows that functions in $W^{n,1}(\mathbb{R}^n)$ are continuous after redefinition on a set of measure zero.)

- The Morrey embedding theorem describes continuous embeddings of Sobolev spaces into appropriate Hölder spaces.

Theorem (Morrey embedding theorem)

If $\alpha \in (0, n)$ and $n/\alpha < p < \infty$, then

$$W^{\alpha,p}(\mathbb{R}^n) \hookrightarrow \Lambda^\gamma(\mathbb{R}^n) \quad \textit{if} \quad \gamma = \alpha - \frac{n}{p}.$$

- Indeed, if $f \in W^{\alpha,p}$ with $\alpha \in (0, n)$ and $n/\alpha < p < \infty$, then

$$\|P_j f\|_{L^\infty} \lesssim 2^{j\frac{n}{p}} \|P_j f\|_{L^p} \lesssim 2^{j\frac{n}{p}} 2^{-j\alpha} \|f\|_{W^{k,p}}$$

for all $j \geq 0$, where the first inequality is the Bernstein inequality from Homework 4, and the second inequality is by the square function characterization of $W^{\alpha,p}$.

Functions of Bounded Mean Oscillation (BMO)

- Let $\alpha \in (0, n)$. The Sobolev and Morrey embedding theorems does not say anything about $W^{\alpha,p}(\mathbb{R}^n)$ if $p = n/\alpha$.
- Indeed $W^{\alpha,\frac{n}{\alpha}}(\mathbb{R}^n)$ does not embed into L^∞.
- To obtain a positive result, we need to introduce the space of functions of bounded mean oscillations (BMO).
- Let f be a locally integrable function on $\mathbb{R}^n$. Define the sharp maximal function $M^\sharp$ by

$$M^\sharp f(x) = \sup_{x \in B} \fint_B |f(y) - f_B| dy, \quad x \in \mathbb{R}^n$$

  where the supremum is over all balls B containing x, and $f_B := \fint_B f$ (recall $\fint_B = \frac{1}{|B|} \int_B$; all balls have positive and finite radius by convention).
- We say that a locally integrable function f on $\mathbb{R}^n$ is in BMO, if

$$\|f\|_{BMO} := \|M^\sharp f\|_{L^\infty(\mathbb{R}^n)} < +\infty.$$

- Note that $\|f\|_{BMO}$ is only a seminorm; $\|f\|_{BMO} = 0$ if and only if f is constant.
- We will see shortly that every BMO function defines a tempered distribution on $\mathbb{R}^n$; thus we usually think of BMO as a subspace of the quotient space $\mathcal{S}'(\mathbb{R}^n)/\{\text{constants}\}$ (and then $\|f\|_{BMO}$ becomes a norm on this quotient).
- Clearly all L^∞ functions on $\mathbb{R}^n$ are in BMO.
- But BMO is larger than L^∞: e.g. $\log|x| \in$ BMO but $\notin L^\infty$.
- Fortunately BMO is not much larger than L^∞: one can prove that if $f \in BMO$, then for every $\varepsilon > 0$, we have

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|)^{n+\varepsilon}} dx < \infty$$

  (which is certainly true if $f \in L^\infty$). This also shows BMO functions define elements in $\mathcal{S}'(\mathbb{R}^n)$.
- Also, BMO scales the same way as L^∞: if $f(x) \in BMO$, then so is $f(\lambda x)$ for any $\lambda > 0$, with the same BMO norm.
- Indeed BMO will act as a substitute for L^∞ for many purposes in harmonic analysis, as the following theorem indicates.

Singular integrals map L^∞ into BMO

Theorem

Let T be a bounded linear operator on $L^2(\mathbb{R}^n)$. Suppose there exists a locally L^∞ function K_0 on $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \colon x \neq y\}$, such that

$$Tf(x) = \int_{\mathbb{R}^n} f(y) K_0(x,y) dy$$

for every $f \in L^1(\mathbb{R}^n)$ with compact support, and a.e. $x \notin supp(f)$. Suppose in addition that

$$\sup_{(x,x_0)\in\mathbb{R}^n\times\mathbb{R}^n} \int_{|y-x_0|\geq 2|x-x_0|} |K_0(x,y) - K_0(x_0,y)| dy \leq C.$$

Then for every bounded and compactly supported function f on $\mathbb{R}^n$, we have

$$\|Tf\|_{BMO} \lesssim \|f\|_{L^\infty}.$$

Furthermore, T can be extended as a continuous linear map from L^∞ to BMO.

- The proof of the theorem uses the following lemma:

Lemma

Let f be a locally integrable function on $\mathbb{R}^n$. Suppose there exists a constant A, such that for every ball B in $\mathbb{R}^n$, there exists a constant c_B satisfying

$$\fint_B |f(y) - c_B| dy \leq A.$$

Then $f \in BMO$, and $\|f\|_{BMO} \leq 2A$.

- Indeed, for every ball B, the triangle inequality shows that $|f_B - c_B| \leq A$, so

$$\fint_B |f(y) - f_B| dy \leq \fint_B |f(y) - c_B| dy + |f_B - c_B| \leq 2A.$$

This shows $\|M^\sharp f\|_{L^\infty} \leq 2A$.

- To prove the theorem, let B be any ball in $\mathbb{R}^n$. Let x_0 be the center of the ball. Let B^* for the ball centered at x_0, but twice the radius of B.
- For every bounded and compactly supported f on $\mathbb{R}^n$, let $f_1 = f\chi_{B^*}$, $f_2 = f\chi_{\mathbb{R}^n \setminus B^*}$ so that $f = f_1 + f_2$.
- Then $Tf_1 \in L^2(\mathbb{R}^n)$ (since $f_1 \in L^2(\mathbb{R}^n)$), and

$$\fint_B |Tf_1(x)|dy \leq \left(\fint_B |Tf_1(x)|^2 dx \right)^{1/2} \lesssim |B|^{-1/2} \|f_1\|_{L^2} \leq \|f\|_{L^\infty}.$$

- Also, since $Tf_2(x) = \int_{\mathbb{R}^n \setminus B^*} f(y) K_0(x,y) dy$ for $x \in B$, if we define $c_B = \int_{\mathbb{R}^n \setminus B^*} f(y) K_0(x_0, y) dy$, then

$$\begin{aligned}\fint_B |Tf_2(x) - c_B| dx &\leq \fint_B \int_{\mathbb{R}^n \setminus B^*} |f(y)||K_0(x,y) - K_0(x_0,y)| dy dx \\ &\leq C\|f\|_{L^\infty}.\end{aligned}$$

- Thus $\fint_B |Tf(x) - c_B| dx \lesssim \|f\|_{L^\infty}$, so $\|f\|_{BMO} \lesssim \|f\|_{L^\infty}$.

- Now given $f \in L^\infty$ on $\mathbb{R}^n$ (not necessarily compactly supported any more), we need to define $Tf(x)$ for a.e. $x \in \mathbb{R}^n$ (modulo constants).
- To do so, let B_0 be a ball centered at the origin. Let B_0^* be the ball centered at the origin and twice the radius of B_0.
- Let $f_1 = f\chi_{B_0^*}$, $f_2 = f\chi_{\mathbb{R}^n \setminus B_0^*}$ so that $f = f_1 + f_2$.
- $Tf_1(x)$ is defined a.e. $x \in B_0$, since $f_1 \in L^2(\mathbb{R}^n)$ and T is bounded on $L^2(\mathbb{R}^n)$.
- For $x \in B_0$, let $Tf_2(x) := \int_{\mathbb{R}^n \setminus B_0^*} f(y)[K_0(x,y) - K_0(0,y)]dy$.
- Then define $Tf(x) = Tf_1(x) + Tf_2(x)$ for a.e. $x \in B_0$.
- This definition depends on the choice of B_0, but if $\tilde{B}_0$ is a ball centered at the origin that contains B_0, then for a.e. $x \in B_0$, the two definitions differ only by $\int_{\tilde{B}_0^* \setminus B_0^*} f(y)K_0(0,y)dy$, which is a constant independent of $x \in B_0$.
- Thus $Tf \in \mathcal{S}'(\mathbb{R}^n)/\{\text{constants}\}$. The earlier argument shows readily that $Tf \in BMO$, with $\|Tf\|_{BMO} \lesssim \|f\|_{L^\infty}$.

- Similarly, we have the following mapping properties of the Riesz potential $\mathcal{I}_\alpha$ into BMO.

Theorem

Let $\alpha \in (0, n)$. For every bounded and compactly supported function f on $\mathbb{R}^n$, we have

$$\|\mathcal{I}_\alpha f\|_{BMO} \lesssim \|f\|_{L^{n/\alpha}}.$$

Furthermore, $\mathcal{I}_\alpha$ can be extended as a continuous linear map from L^∞ to BMO.

- To prove the theorem, let $\alpha \in (0, n)$, and $f \in L^{n/\alpha}$ on $\mathbb{R}^n$.
- Let B_0 be a ball centered at the origin. Let B_0^* be the ball centered at the origin and twice the radius of B_0.
- Let $f_1 = f\chi_{B_0^*}$, $f_2 = f\chi_{\mathbb{R}^n \setminus B_0^*}$ so that $f = f_1 + f_2$.
- $\mathcal{I}_\alpha f_1(x)$ is defined a.e. $x \in B_0$, since $f_1 \in L^p(\mathbb{R}^n)$ for all $1 < p < n/\alpha$, and $\mathcal{I}_\alpha \colon L^p(\mathbb{R}^n) \to L^{\frac{np}{n-\alpha p}}(\mathbb{R}^n)$ for such p's.
- For $x \in B_0$, let

$$\mathcal{I}_\alpha f_2(x) = c_{\alpha,n} \int_{\mathbb{R}^n \setminus B_0^*} f(y) \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|y|^{n-\alpha}} \right) dy.$$

  Hölder's inequality shows that $\mathcal{I}_\alpha f_2 \in L^\infty(B_0)$.
- Now define $\mathcal{I}_\alpha f(x) = \mathcal{I}_\alpha f_1(x) + \mathcal{I}_\alpha f_2(x)$ for a.e. $x \in B_0$.
- This definition depends on the choice of B_0, but if $\tilde{B}_0$ is a ball centered at the origin that contains B_0, then for a.e. $x \in B_0$, the two definitions differ only by $\int_{\tilde{B}_0{}^* \setminus B_0^*} f(y)|y|^{-(n-\alpha)} dy$, which is a constant independent of $x \in B_0$.

- The earlier argument shows readily that $\mathcal{I}_\alpha f \in BMO$, with $\|\mathcal{I}_\alpha f\|_{BMO} \lesssim \|f\|_{L^{n/\alpha}}$; indeed for any ball B in $\mathbb{R}^n$, we have

$$\fint_B |\mathcal{I}_\alpha(f\chi_{B^*})(x)| dx \lesssim \left(\fint_B |\mathcal{I}_\alpha(f\chi_{B^*})(x)|^{p^*} dx \right)^{1/p^*}$$
$$\lesssim |B|^{-\frac{1}{p^*}} \|f\chi_{B^*}\|_{L^p}$$
$$\lesssim |B|^{-\frac{1}{p^*}} |B|^{\frac{\alpha}{n} - \frac{1}{p}} \|f\|_{L^{n/\alpha}}$$
$$= \|f\|_{L^{n/\alpha}}$$

where p is any exponent satisfying $1 < p < n/\alpha$, and $\frac{1}{p^*} = \frac{1}{p} - \frac{\alpha}{n}$.

- Furthermore, if B_0 is a sufficiently large ball centered at the origin so that B_0^* contains B^*, then defining $\mathcal{I}_\alpha(f\chi_{\mathbb{R}^n\setminus B^*})$ on B_0 as before, and letting

$$c_B := c_{n,\alpha}\int_{\mathbb{R}^n\setminus B^*} f(y)\left(\frac{1}{|x_0-y|^{n-\alpha}}-\frac{1}{|y|^{n-\alpha}}\right)dy$$

where x_0 is the center of B, we have

$$\begin{aligned}
&\fint_B |\mathcal{I}_\alpha(f\chi_{\mathbb{R}^n\setminus B^*})(x)-c_B|dx\\
&\lesssim \|f\|_{L^{n/\alpha}}\fint_B\left(\int_{\mathbb{R}^n\setminus B^*}\left|\frac{1}{|x-y|^{n-\alpha}}-\frac{1}{|x_0-y|^{n-\alpha}}\right|^{\frac{n}{n-\alpha}}dy\right)^{\frac{n-\alpha}{n}}dx\\
&\lesssim \|f\|_{L^{n/\alpha}}\fint_B\left(\int_{\mathbb{R}^n\setminus B^*}\frac{1}{|x_0-y|^{n+\frac{n}{n-\alpha}}}dy\right)^{\frac{n-\alpha}{n}}|x-x_0|dx\\
&\lesssim \|f\|_{L^{n/\alpha}}.
\end{aligned}$$

- Together we see that $\|\mathcal{I}_\alpha f\|_{BMO}\lesssim \|f\|_{L^{n/\alpha}}$.

- To summarize, for $f \in L^{n/\alpha}(\mathbb{R}^n)$, we have defined $\mathcal{I}_\alpha f$ as an element of BMO, and hence as an element of the quotient space $\mathcal{S}'(\mathbb{R}^n)/\{\text{constants}\}$.
- Note that for $f \in L^p(\mathbb{R}^n)$, $1 \leq p < n/\alpha$, we had defined $\mathcal{I}_\alpha f$ as L^{p^*} or weak-L^{p^*} functions in Chapter 3, and hence as elements of $\mathcal{S}'(\mathbb{R}^n)$.
- For $f \in L^{n/\alpha} \cap L^p(\mathbb{R}^n)$ for some $1 \leq p < n/\alpha$, this old definition agrees with the new one above when tested against Schwartz functions whose integrals are zero.

Sobolev embeddings into BMO

- The previous theorem allows us to prove embeddings of Sobolev spaces into BMO.

Theorem

Let $\alpha \in (0, n)$. Then

$$W^{\alpha, \frac{n}{\alpha}}(\mathbb{R}^n) \hookrightarrow BMO.$$

- To see this, let $f \in W^{\alpha, \frac{n}{\alpha}}(\mathbb{R}^n)$. Write

$$f = P_0 f + \mathcal{I}_\alpha (I - P_0)(-\Delta)^{\alpha/2} \mathcal{J}_\alpha [(I - \Delta)^{\alpha/2} f].$$

  Since $\|P_0 f\|_{L^\infty} \lesssim \|f\|_{L^{n/\alpha}}$, $(I - \Delta)^{\alpha/2} f \in L^{n/\alpha}$, $(I - P_0)(-\Delta)^{\alpha/2} \mathcal{J}_\alpha$ preserves $L^{n/\alpha}$, and $\mathcal{I}_\alpha \colon L^{n/\alpha} \to BMO$, we see that $f \in BMO$, as desired.
- Indeed more precise estimates are possible; see Homework 5 for a discussion of the Moser-Trudinger inequality, which is important in conformal geometry.

The John-Nirenberg inequality for BMO

- We close this lecture by stating the John-Nirenberg inequality.

Theorem (John-Nirenberg)

(a) *There exists constants C_1, C_2 depending only on n, such that for any BMO function f on $\mathbb{R}^n$, and any cube $Q \subset \mathbb{R}^n$, we have*

$$\frac{|\{x \in Q : |f(x) - f_Q| > \lambda\}|}{|Q|} \leq C_1 e^{-\frac{C_2 \lambda}{\|f\|_{BMO}}}$$

for all $\lambda > 0$. Here $f_Q := \fint_Q f$ is the average of f on Q.

(b) *For any $p \in (1, \infty)$, there exists a constant $C_{n,p}$, such that*

$$\sup_Q \left(\fint_Q |f(y) - f_Q|^p dy \right)^{1/p} \leq C_{n,p} \|f\|_{BMO}$$

for every BMO function f on $\mathbb{R}^n$, where the supremum is over all cubes $Q \subset \mathbb{R}^n$; indeed one may take $C_{n,p}^p \leq C_1 \Gamma(p+1) C_2^{-p}$ where C_1, C_2 are as in part (a).

- Note that in the inequality in part (b), the right hand side is certainly bounded by the left hand side, by Hölder's inequality. The inequality in part (b) is thus sometimes called a reverse Hölder inequality.
- The bound for the constant in part (b) gives us the following corollary:

Corollary

There exists constants $c, C > 0$ depending only on n, such that for every BMO function f on $\mathbb{R}^n$ and every cube $Q \subset \mathbb{R}^n$, we have

$$\fint_Q \exp\left(\frac{c|f(y) - f_Q|}{\|f\|_{BMO}}\right) dy \leq C.$$

- The proof of part (a) of the theorem can be achieved by iteratively performing Calderón-Zygmund decomposition.
- Part (b) of the theorem and the corollary then follows easily.
- For details of the proofs, see Homework 5.

- We remark though that in the special case when $f \in BMO(\mathbb{R})$ is non-negative and decreasing on $(0,1)$, one can easily see that
$$f(x) = O(\log(1/x)) \quad \text{as } x \to 0^+,$$
and hence $\exp(cf)$ is integrable on $(0,1)$ for all sufficiently small c; indeed if $I_j = [2^{-j-1}, 2^{-j}]$ then $f_{I_j} - f_{I_{j-1}} \lesssim \|f\|_{BMO}$, so
$$f_{I_j} \leq f_{I_0} + Cj\|f\|_{BMO}$$
for all $j \geq 1$, which implies the desired pointwise bound for f given the monotonicity of f. With a bit of care chasing through this argument, one can also see that
$$\int_{(0,1)} \exp\left(\frac{c|f(y) - f_{(0,1)}|}{\|f\|_{BMO}}\right) dy < \infty,$$
which is part of the claim of the corollary on the previous slide.
- The John-Nirenberg inequality (in particular, part (b) of the theorem) will be used in the proof of the Carleson embedding theorem, which will in turn play a pivotal role in the proof of the T(1) theorem in Lecture 7.