Topics in Harmonic Analysis
Lecture 8: Interpolation

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Real interpolation

- In this lecture we study real and complex interpolation.
- These are methods of deducing boundedness of certain linear or quasi-additive operators on certain "intermediate" function spaces, from the boundedness of these operators on some other "endpoint" function spaces.
- We begin with the real method of interpolation, following Marcinkiewicz.
- We have already seen a version of it in the study of maximal functions and singular integrals in Lectures 3 and 4.
- We will sometimes encounter Lebesgue spaces L^p with $p < 1$, and the statement of Marcinkiewicz interpolation theorem is best formulated using Lorentz spaces $L^{p,r}$.
- We introduce these in the next few slides.

Lebesgue spaces for $p < 1$

- Let (X, μ) be a measure space, and $f \colon X \to \mathbb{C}$ be measurable.
- For $p \in (0, 1)$, we still say $f \in L^p$ if

$$\|f\|_{L^p} := \left(\int_X |f|^p d\mu \right)^{1/p} < \infty.$$

- Note that $\| \cdot \|_{L^p}$ does not define a norm when $p \in (0, 1)$; the triangle inequality is not satisfied.
- The following is often a useful substitute:

$$\|f + g\|_{L^p}^p \leq \|f\|_{L^p}^p + \|g\|_{L^p}^p$$

which holds for all $f, g \in L^p$, $p \in (0, 1]$.

- From this we deduce a quasi-triangle inequality: for all $p \in (0, 1)$, there exists some finite constant C_p such that

$$\|f + g\|_{L^p} \leq C_p(\|f\|_{L^p} + \|g\|_{L^p})$$

for all $f, g \in L^p$.

Lorentz spaces $L^{p,r}$

- Next we introduce Lorentz spaces.
- Let (X,μ) be a measure space, and $f\colon X \to \mathbb{C}$ be measurable.
- Let $p \in (0,\infty)$, $r \in (0,\infty]$. f is said to be in the Lorentz space $L^{p,r}$, if $|||f|||_{L^{p,r}} < \infty$, where

$$|||f|||_{L^{p,r}} := \left(p \int_0^\infty \left[\alpha\mu\{|f| > \alpha\}^{1/p} \right]^r \frac{d\alpha}{\alpha} \right)^{1/r} \quad \text{if } r \in (0,\infty);$$

$$|||f|||_{L^{p,r}} := \sup_{\alpha>0} \left[\alpha\,\mu\{|f| > \alpha\}^{1/p} \right] \quad \text{if } r = \infty.$$

- Note that $L^{p,\infty}$ is the weak-L^p space introduced in Lecture 3.
- By convention, $L^{\infty,\infty}$ is L^∞, and $L^{\infty,r}$ is undefined for $r < \infty$.
- Observe also $|||f|||_{L^{p,p}} = \|f\|_{L^p}$ by Fubini for all $p \in (0,\infty]$.
- It is often convenient to note that

$$|||f|||_{L^{p,r}} \simeq \|2^k \mu\{|f| > 2^k\}^{1/p}\|_{\ell^r(\mathbb{Z})}$$

for all measurable f and all $p \in (0,\infty)$, $r \in (0,\infty]$.

- In general $|||\cdot|||_{L^{p,r}}$ defines only a quasi-norm on $L^{p,r}$, and not a norm. In other words, the triangle inequality is not satisfied, but we have

$$|||f+g|||_{L^{p,r}} \leq C_{p,r} \left(|||f|||_{L^{p,r}} + |||g|||_{L^{p,r}} \right)$$

for some finite constant $C_{p,r} \geq 1$.
- But $L^{p,r}$ does admit a comparable norm if $p \in (1,\infty)$ and $r \in [1,\infty]$; indeed when $p \in (1,\infty)$ and $r \in (1,\infty]$, $L^{p,r}$ is the dual space of $L^{p',r'}$, so it admits a dual norm

$$\|f\|_{L^{p,r}} := \sup \left\{ \left| \int_X fg \, d\mu \right| : |||g|||_{L^{p',r'}} \leq 1 \right\}.$$

The same construction works when $p \in (1,\infty)$ and $r = 1$. See Homework 8 for details, and Stein and Weiss' *Introduction to Fourier Analysis*, Chapter V.3, for an alternative approach of norming $L^{p,r}$.

- To formulate the Marcinkiewicz interpolation theorem, let (X, μ), (Y, ν) be measure spaces.
- Let T be an operator defined on a subspace $\mathrm{Dom}(T)$ of measurable functions on X, that maps each element in $\mathrm{Dom}(T)$ to a measurable function on Y.
- We say T is subadditive if

$$|T(f+g)| \leq |Tf| + |Tg|$$

for all $f, g \in \mathrm{Dom}(T)$.
- Suppose $\mathrm{Dom}(T)$ is stable under truncations, i.e. if $f \in \mathrm{Dom}(T)$ then $f\chi_E$ is in $\mathrm{Dom}(T)$ for all measurable subsets E of X, where χ_E is the characteristic function of E.
- Let $p, q \in (0, \infty]$. If $p \neq \infty$, then we say that T is of restricted weak-type (p, q), if

$$\|Tf\|_{L^{q,\infty}} \lesssim \|f\|_{L^{p,1}} \quad \text{for all } f \in \mathrm{Dom}(T) \cap L^{p,1};$$

if $p = \infty$, then we say that T is of restricted weak-type (p, q), if the same holds with $L^{p,1}$ replaced by L^∞.

Theorem (Marcinkiewicz interpolation theorem)

Let $p_0, p_1, q_0, q_1 \in (0, \infty]$ with $p_0 \neq p_1$ and $q_0 \neq q_1$. Let p, q be such that

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

for some $\theta \in (0,1)$. If a subadditive operator T is of restricted weak-types (p_0, q_0) and (p_1, q_1), then for any $r \in (0, \infty]$, we have

$$\|Tf\|_{L^{q,r}} \lesssim \|f\|_{L^{p,r}}$$

for all f in $Dom(T) \cap L^{p,r}$; in particular, if $p \leq q$, then $\|Tf\|_{L^q} \lesssim \|f\|_{L^p}$ for all f in $Dom(T) \cap L^p$.

- In applications usually we have both $p_0 \leq q_0$ and $p_1 \leq q_1$, from which it follows that $p \leq q$.

- Here we mention a related observation:

Proposition

Let $p_0, p_1, q_0, q_1 \in (0, \infty]$ *with* $p_0 = p_1$ *and* $q_0 \neq q_1$. *Let* p, q *be as in the previous theorem with* $\theta \in (0,1)$. *If a subadditive operator* T *is of weak-types* (p_0, q_0) *and* (p_1, q_1) *(not just restricted weak-types), then for all* $r \in (0, \infty]$, *we have*

$$\|Tf\|_{L^{q,r}} \lesssim \|f\|_{L^{p,r}}$$

for all f *in* $Dom(T) \cap L^{p,r}$.

- The proposition follows just from the inclusions $L^{p,r} \subseteq L^{p,\infty}$ and $L^{q_0,\infty} \cap L^{q_1,\infty} \subseteq L^{q,r}$. But the condition $q_0 \neq q_1$ is crucial.
- Combining the theorem (the case where $p_0 \leq q_0$ and $p_1 \leq q_1$) with the proposition, we obtain the following corollary:

Corollary (weak-type case of Marcinkiewicz interpolation)

Let $p_0, p_1, q_0, q_1 \in (0, \infty]$ *with* $q_0 \neq q_1$, $p_0 \leq q_0$ *and* $p_1 \leq q_1$. *Let* p, q *be such that*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \textit{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

for some $\theta \in (0, 1)$. *If a subadditive operator* T *is of weak-types* (p_0, q_0) *and* (p_1, q_1), *then*

$$\|Tf\|_{L^q} \lesssim \|f\|_{L^p}$$

for all f *in* $Dom(T) \cap L^p$.

- We now turn to the proof of the theorem.
- We will only prove the case when p_0, p_1, q_0, q_1 are all finite; the cases where one of the p_i's is infinite, and/or where one of the q_i's is infinite, is left as an exercise (see Homework 8).
- Let $p_0, p_1, q_0, q_1 \in (0, \infty)$ with $p_0 \neq p_1$ and $q_0 \neq q_1$. Let $\theta \in (0, 1)$, and define p, q as in the theorem.
- It will be convenient to write

$$\alpha = p\left(\frac{1}{p_0} - \frac{1}{p_1}\right), \quad \beta = q\left(\frac{1}{q_0} - \frac{1}{q_1}\right),$$

$$x_0 = q_0\theta, \quad x_1 = -q_1(1-\theta).$$

- We have $\alpha \neq 0$ and $\beta \neq 0$ by assumption.
- Let $r \in (0, \infty)$, $f \in L^{p,r}$ with $|||f|||_{L^{p,r}} = 1$. We will show that

$$\|Tf\|_{L^{q,r}} \lesssim 1.$$

- Decompose
$$f = \sum_{k \in \mathbb{Z}} f_k$$
where $f_k = f\chi_{2^k < |f| \leq 2^{k+1}}$. Write $W_k = \mu(\mathrm{supp} f_k)$, so that
$$\sum_{k \in \mathbb{Z}} 2^{kr} W_k^{r/p} \lesssim 1.$$
- For $k \in \mathbb{Z}$, we define
$$a_k = \sum_{\ell \in \mathbb{Z}} 2^{-|k-\ell|\varepsilon} 2^{\ell r} W_\ell^{r/p}$$
where $\varepsilon > 0$ is a small parameter to be determined. Then
$$W_k \leq 2^{-kp} a_k^{p/r} \quad \text{for all } k \in \mathbb{Z} \quad \text{and} \quad \sum_{k \in \mathbb{Z}} a_k \lesssim 1,$$
(and these would also hold if we had simply defined a_k to be $2^{kr} W_k^{r/p}$), but the additional sup over ℓ in the definition of a_k guarantees that a_k does not vary too rapidly, in the sense that
$$a_k \leq 2^{|k-\ell|\varepsilon} a_\ell \quad \text{for all } k, \ell \in \mathbb{Z}.$$

- In particular, since $a_\ell \lesssim 1$ for all ℓ, taking $\ell \simeq \frac{j\beta}{\alpha}$, we have

$$a_k^{\frac{\alpha-\beta}{r}} \lesssim 2^{|j\beta - k\alpha| C\varepsilon}$$

for some finite constant $C = C_{\alpha,\beta,r}$.

- For $k, j \in \mathbb{Z}$, let

$$c_{k,j} := 2^{-|j\beta - k\alpha|\varepsilon}.$$

- Then since $\alpha \neq 0$, $\sum_{k\in\mathbb{Z}} c_{k,j} \lesssim_\varepsilon 1$, so by subadditivity of T,

$$\mu\{|Tf| > 2^j\} \leq \sum_{k\in\mathbb{Z}} \mu\{|Tf_k| \gtrsim_\varepsilon c_{k,j} 2^j\},$$

which by the restricted weak-type properties of T is bounded above by

$$\lesssim_\varepsilon \sum_{k\in\mathbb{Z}} \min_{i=0,1} \left(c_{k,j}^{-1} 2^{-j} 2^k W_k^{1/p_i} \right)^{q_i}.$$

(We used the finiteness of p_0, p_1, q_0, q_1 here.)

- Hence to show that $\|Tf\|_{L^{q,r}} \lesssim 1$, it suffices to show that

$$\sum_{j\in\mathbb{Z}} 2^{jr} \left[\sum_{k\in\mathbb{Z}} \min_{i=0,1} \left(c_{k,j}^{-1} 2^{-j} 2^{k} W_k^{1/p_i} \right)^{q_i} \right]^{\frac{r}{q}} \lesssim 1.$$

- Now using $W_k \le 2^{-kp} a_k^{p/r}$, we just need to show

$$\sum_{j\in\mathbb{Z}} \left[\sum_{k\in\mathbb{Z}} \min_{i=0,1} \left(c_{k,j}^{-1} 2^{jq\left(\frac{1}{q_i}-\frac{1}{q}\right)} 2^{-kp\left(\frac{1}{p_i}-\frac{1}{p}\right)} a_k^{\frac{p}{rp_i}} \right)^{q_i} \right]^{\frac{r}{q}}.$$

- Since

$$pq_i\left(\frac{1}{p_i}-\frac{1}{p}\right) = \alpha x_i \quad \text{and} \quad qq_i\left(\frac{1}{q_i}-\frac{1}{q}\right) = \beta x_i,$$

the above is just

$$\sum_{j\in\mathbb{Z}} \left[\sum_{k\in\mathbb{Z}} \min_{i=0,1} \left(c_{k,j}^{-q_i} 2^{(j\beta-k\alpha)x_i} a_k^{\frac{pq_i}{rp_i}} \right) \right]^{\frac{r}{q}}.$$

- Now factor our $a_k^{q/r}$ from the minimum in the sum. Since

$$\frac{pq_i}{p_i} - q = pq_i\left(\frac{1}{p_i} - \frac{1}{p}\right) + q_i - q = \alpha x_i - qq_i\left(\frac{1}{q_i} - \frac{1}{q}\right)$$

which equals $(\alpha - \beta)x_i$, the above is just

$$\sum_{j\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} a_k^{\frac{q}{r}} \min_{i=0,1}\left(c_{k,j}^{-q_i} 2^{(j\beta - k\alpha)x_i} a_k^{\frac{(\alpha-\beta)x_i}{r}}\right)\right]^{\frac{r}{q}}.$$

- In view of our earlier bound for $a_k^{\frac{\alpha-\beta}{r}}$ and $c_{k,j}$, this is bounded by

$$\sum_{j\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}} a_k^{\frac{q}{r}} \min_{i=0,1}\left(2^{(j\beta - k\alpha)x_i} 2^{|j\beta - k\alpha|c\varepsilon}\right)\right]^{\frac{r}{q}} \qquad (1)$$

for some finite constant c.

- We now choose $\varepsilon > 0$ sufficiently small, so that

$$c\varepsilon < \min\{|x_0|, |x_1|\}.$$

- If $\frac{r}{q} \leq 1$, then we use $[\sum_k \dots]^{r/q} \leq \sum_k [\dots]^{r/q}$, and bound (1) by

$$\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} a_k \min_{i=0,1} \left(2^{(j\beta-k\alpha)x_i} 2^{|j\beta-k\alpha|c\varepsilon} \right)^{\frac{r}{q}}.$$

Since x_0, x_1 are non-zero, of opposite signs, and $\beta \neq 0$, in view of our earlier choice of ε, we have

$$\sum_{j\in\mathbb{Z}} \min_{i=0,1} \left(2^{(j\beta-k\alpha)x_i} 2^{|j\beta-k\alpha|c\varepsilon} \right)^{\frac{r}{q}} \lesssim 1$$

uniformly in k, so

$$(1) \lesssim \sum_{k\in\mathbb{Z}} a_k \lesssim 1.$$

- If $\frac{r}{q} \geq 1$, we use the observation that

$$\sum_{k\in\mathbb{Z}} \min_{i=0,1} \left(2^{(j\beta-k\alpha)x_i} 2^{|j\beta-k\alpha|c\varepsilon}\right) \lesssim 1$$

uniformly in j. Jensen's inequality then shows

$$\left[\sum_{k\in\mathbb{Z}} a_k^{\frac{q}{r}} \min_{i=0,1} \left(2^{(j\beta-k\alpha)x_i} 2^{|j\beta-k\alpha|c\varepsilon}\right)\right]^{\frac{r}{q}} \lesssim \sum_{k\in\mathbb{Z}} a_k \min_{i=0,1} \left(2^{(j\beta-k\alpha)x_i} 2^{|j\beta-k\alpha|c\varepsilon}\right),$$

which we then sum over j to yield

$$(1) \lesssim \sum_{k\in\mathbb{Z}} a_k \sum_{j\in\mathbb{Z}} \min_{i=0,1} \left(2^{(j\beta-k\alpha)x_i} 2^{|j\beta-k\alpha|c\varepsilon}\right) \lesssim 1.$$

(We used again $\beta \neq 0$ to evaluate the last sum over j.)

- This completes the proof of the Marcinkiewicz interpolation theorem when $r \in (0, \infty)$.
- When $r = \infty$ the proof is easier.
- Indeed, let $f \in L^{p,\infty}$ with $\|f\|_{L^{p,\infty}} = 1$, and let $\lambda > 0$.
- To estimate $\mu\{|Tf| > \lambda\}$, we decompose $f = f_0 + f_1$, where $f_0 = f\chi_{|f|>\gamma}$ and $f_1 = f\chi_{|f|\leq\gamma}$.
- We have

$$\|f_0\|_{L^{p_0,1}} \lesssim \sum_{2^k>\gamma} 2^k \mu\{|f| > 2^k\}^{\frac{1}{p_0}} \lesssim \sum_{2^k>\gamma} 2^k 2^{-\frac{kp}{p_0}} = \gamma^{-p\left(\frac{1}{p_0}-\frac{1}{p}\right)}$$

and similarly

$$\|f_1\|_{L^{p_1,1}} \lesssim \gamma^{-p\left(\frac{1}{p_1}-\frac{1}{p}\right)}.$$

- As a result,

$$\begin{aligned}\mu\{|Tf| > \lambda\} &\le \mu\{|Tf_0| > \lambda/2\} + \mu\{|Tf_1| > \lambda/2\} \\ &\lesssim \lambda^{-q_0}\|f_0\|_{L^{p_0,1}}^{q_0} + \lambda^{-q_1}\|f_1\|_{L^{p_1,1}}^{q_1}\end{aligned}$$

  which is bounded by

$$\begin{aligned}&\lambda^{-q}\left(\lambda^{qq_0\left(\frac{1}{q_0}-\frac{1}{q}\right)}\gamma^{-pq_0\left(\frac{1}{p_0}-\frac{1}{p}\right)} + \lambda^{qq_1\left(\frac{1}{q_1}-\frac{1}{q}\right)}\gamma^{-pq_1\left(\frac{1}{p_1}-\frac{1}{p}\right)}\right) \\ &= \lambda^{-q}\left(\lambda^{\beta x_0}\gamma^{-\alpha x_0} + \lambda^{\beta x_1}\gamma^{-\alpha x_1}\right).\end{aligned}$$

- Choosing $\gamma = \lambda^{\beta/\alpha}$ gives

$$\mu\{|Tf| > \lambda\} \lesssim \lambda^{-q},$$

  as desired.

Complex interpolation

- Next we turn to the complex method of interpolation, following Riesz, Thorin and Stein.
- The key is the following three lines lemma, which is a variant of the maximum principle for holomorphic functions on a strip (whose proof we defer to Homework 8):

Lemma

Let S be the strip $\{0 < Re\, z < 1\}$, and $\overline{S}$ be its closure. Suppose f is a holomorphic function on the strip S that extends continuously to $\overline{S}$. Assume $|f(z)| \leq A_0$ when $Re\, z = 0$, and $|f(z)| \leq A_1$ when $Re\, z = 1$. If there exist $\alpha < 1$, and constants C, c, such that

$$|f(z)| \leq Ce^{ce^{\pi\alpha|z|}}$$

for all $z \in \overline{S}$, then $|f(z)| \leq A_0^{1-Re\, z} A_1^{Re\, z}$ on $\overline{S}$.

- The condition $|f(z)| \leq Ce^{ce^{\pi\alpha|z|}}$ would be satisfied, if say $|f|$ is bounded on the strip.

- To proceed further, if (X, μ) is a measure space, and $p_0, p_1 \in [1, \infty]$, then we denote by $L^{p_0} + L^{p_1}$ the space of all functions f on X such that $f = f_0 + f_1$ for some $f_0 \in L^{p_0}$ and $f_1 \in L^{p_1}$. This can be made a Banach space with norm

$$\|f\|_{L^{p_0}+L^{p_1}} := \inf \left\{ \|f_0\|_{L^{p_0}} + \|f_1\|_{L^{p_1}} : f = f_0 + f_1, f_0 \in L^{p_0}, f_1 \in L^{p_1} \right\}.$$

  Note that L^p embeds continuously into $L^{p_0} + L^{p_1}$ if p is between p_0 and p_1.
- We will also need the Banach space $L^{p_0} \cap L^{p_1}$, where $p_0, p_1 \in [1, \infty]$. Indeed, this is equipped with norm

$$\|g\|_{L^{p_0} \cap L^{p_1}} := \max \left\{ \|g\|_{L^{p_0}}, \|g\|_{L^{p_1}} \right\};$$

  $L^{p_0} \cap L^{p_1}$ embeds continuously into L^p if p is between p_0 and p_1.

Theorem (Riesz-Thorin)

Let (X,μ), (Y,ν) *be measure spaces. Let* $p_0, p_1, q_0, q_1 \in [1,\infty]$, *and* $T\colon (L^{p_0}+L^{p_1})(X) \to (L^{q_0}+L^{q_1})(Y)$ *be a linear operator. Suppose there exist constants* A_0, A_1 *such that*

$$\|Tf\|_{L^{q_0}} \le A_0 \|f\|_{L^{p_0}} \quad \text{for all } f \in L^{p_0}(X),$$

$$\|Tf\|_{L^{q_1}} \le A_1 \|f\|_{L^{p_1}} \quad \text{for all } f \in L^{p_1}(X).$$

Then for any $\theta \in (0,1)$, *we have*

$$\|Tf\|_{L^q} \le A_0^{1-\theta} A_1^{\theta} \|f\|_{L^p}$$

for $f \in L^p(X)$, *where*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

- Before we prove the theorem, recall that a simple function on X is a function of the form

$$\sum_{j=1}^{J} a_j \chi_{E_j}$$

where $J \in \mathbb{N}$, $a_1, \dots, a_J \in \mathbb{C}$ and $E_1, \dots, E_J$ are measurable subsets of X of finite measures.
- Note that if $p \in (0, \infty)$, the set of simple functions on X is dense in L^p (the same is true for $p = \infty$ if in addition X is σ-finite, but we will not need this).
- The key to the proof of the theorem is the following proposition (where as before $S = \{0 < \operatorname{Re} z < 1\}$):

Proposition

Let (X, μ) be a measure space. Let $p_0, p_1 \in (0, \infty]$ and $\theta \in (0, 1)$. Let p be the exponent given by $1/p = (1-\theta)/p_0 + \theta/p_1$. Let f be any simple function on X. Then for any $z \in \overline{S}$, there exists a simple function f_z on X, such that the (vector-valued) map $z \mapsto f_z$ is holomorphic on S, continuous on $\overline{S}$, bounded on $\overline{S}$, and satisfies

$$\|f_z\|_{L^{p_j}} \leq \|f\|_{L^p} \quad \text{when } \mathrm{Re}\, z = j, \text{ for } j = 0, 1,$$

with $f_\theta = f$.

- Indeed, it suffices to take

$$f_z(x) = \frac{f(x)}{|f(x)|} \frac{|f(x)|^{p\left(\frac{1-z}{p_0} + \frac{z}{p_1}\right)}}{\|f\|_{L^p}^{p\left(\frac{1-z}{p_0} + \frac{z}{p_1}\right)}} \|f\|_{L^p}.$$

- To prove the theorem, let $p_0, p_1, q_0, q_1 \in [1, \infty]$, $\theta \in (0, 1)$, and define p, q as in the statement of the theorem.
- Suppose $p \neq \infty$. We claim that it suffices to show that

$$\|Tf\|_{L^q} \leq A_0^{1-\theta} A_1^{\theta} \|f\|_{L^p} \tag{2}$$

  for all simple functions f on X.
- Indeed, then given a general $f \in L^p(X)$, we take a sequence $\{f_n\}$ of simple functions such that $f_n \to f$ in $L^p(X)$ as $n \to \infty$.
- Under the hypothesis of the theorem, the map $T\colon (L^{p_0} + L^{p_1})(X) \to (L^{q_0} + L^{q_1})(Y)$ is continuous.
- By continuity of the inclusion of $L^p(X)$ into $(L^{p_0} + L^{p_1})(X)$, it follows that $Tf_n \to Tf$ in $(L^{q_0} + L^{q_1})(Y)$.
- But by (2), $\{Tf_n\}$ is Cauchy in $L^q(Y)$, so it converges in L^q.
- Since convergence in L^q implies convergence in $L^{q_0} + L^{q_1}$, we see that $Tf \in L^q(Y)$, and that $Tf_n \to Tf$ in $L^q(Y)$, so (2) holds for this general $f \in L^p(X)$ as well.

- Let now f be a simple function on X. We establish (2) for f.
- We consider two cases, namely $q \neq 1$ and $q = 1$.
- Assume first $q \neq 1$. By density of simple functions in $L^{q'}(Y)$, it suffices to show that

$$\left| \int_Y Tf \cdot g d\nu \right| \leq A_0^{1-\theta} A_1^{\theta} \|f\|_{L^p} \|g\|_{L^{q'}} \tag{3}$$

for all simple functions g on Y.

- So fix two simple functions f and g on X and Y respectively.
- We apply the earlier proposition to f, p_0, p_1, θ and g, q_0', q_1', θ, and obtain a holomorphic family f_z and g_z, where the key properties are that

$$\|f_z\|_{L^{p_j}} \leq \|f\|_{L^p} \quad \text{and} \quad \|g_z\|_{L^{q_j'}} \leq \|g\|_{L^{q'}}$$

when $\operatorname{Re} z = j$, for $j = 0, 1$, and that $f_\theta = f$, $g_\theta = g$.

- Let now

$$F(z) = \int_Y Tf_z \cdot g_z d\nu.$$

  We see that $F(z)$ is holomorphic on the strip S, continuous on $\overline{S}$, and bounded on $\overline{S}$. Also, the assumed bound of T on L^{p_0} and L^{p_1} shows that

$$|F(z)| \leq A_j \|f\|_{L^p} \|g\|_{L^{q'}} \quad \text{when } \operatorname{Re} z = j, \text{ for } j = 0, 1.$$

- So the three lines lemma imply $|F(\theta)| \leq A_0^{1-\theta} A_1^{\theta} \|f\|_{L^p} \|g\|_{L^{q'}}$, which is the desired conclusion (3) since $f_\theta = f$ and $g_\theta = g$.
- On the other hand, if $q = 1$, we will show directly that

$$\left| \int_Y Tf \cdot g d\nu \right| \leq A_0^{1-\theta} A_1^{\theta} \|f\|_{L^p} \|g\|_{L^{q'}} \tag{4}$$

  for all $g \in L^{q'}(Y)$.
- So fix a simple function f on X, and a general $g \in L^{q'}(Y)$.
- Note that since $q = 1$, we have $q_0 = q_1 = q$, so we already have $g \in (L^{q_0{}'} \cap L^{q_1{}'})(Y)$.

- We apply the earlier proposition to f, p_0, p_1, θ only, and obtain a holomorphic family f_z; then consider

$$F(z) = \int_Y Tf_z \cdot g d\nu$$

- Since $g \in (L^{q_0{}'} \cap L^{q_1{}'})(Y)$, our assumptions imply that $F(z)$ is holomorphic on the strip S, continuous on $\overline{S}$, and bounded on $\overline{S}$. Also, the assumed bound of T on L^{p_0} and L^{p_1} shows that

$$|F(z)| \leq A_j \|f\|_{L^p} \|g\|_{L^{q'}} \quad \text{when } \mathrm{Re}\, z = j, \text{ for } j = 0, 1.$$

- So the three lines lemma imply $|F(\theta)| \leq A_0^{1-\theta} A_1^{\theta} \|f\|_{L^p} \|g\|_{L^{q'}}$, which is the desired conclusion (4) since $f_\theta = f$.
- This completes the proof of the theorem when $p \neq \infty$.
- When $p = \infty$, we simply show directly that

$$\|Tf\|_{L^q} \leq A_0^{1-\theta} A_1^{\theta} \|f\|_{L^p} \tag{5}$$

for all $f \in L^p(X)$.

- Indeed, let f be a general function in $L^p(X)$. Then since $p = \infty$, we have $p_0 = p_1 = p$, so we have $f \in (L^{p_0} \cap L^{p_1})(X)$.
- If $q \neq 1$, then we show that

$$\left| \int_Y Tf \cdot g d\nu \right| \leq A_0^{1-\theta} A_1^{\theta} \|f\|_{L^p} \|g\|_{L^{q'}}$$

  for all simple functions g on Y, by considering $\int_Y Tf \cdot g_z d\nu$ for a suitable holomorphic extension of the simple function g; if $q = 1$, we show that the same holds for all $g \in L^{q'}(Y)$ directly.
- This completes the proof of the Riesz-Thorin theorem. (The cases $p = \infty$ or $q = 1$ would not require a separate treatment if we assume both X and Y are σ-finite.)
- Coming up next is a remarkably useful observation of Stein, namely that the Riesz-Thorin theorem also works for an analytic family of operators.
- As before, denote by S the strip $\{0 < \operatorname{Re} z < 1\}$, and $\overline{S}$ the closure of S.

Theorem (Stein)

Let (X,μ), (Y,ν) be measure spaces. Let $p_0, p_1, q_0, q_1 \in [1,\infty]$. Suppose $\{T_z\}_{z\in\overline{S}}$ is a family of bounded linear operators from $(L^{p_0}\cap L^{p_1})(X)$ to $(L^{q_0}+L^{q_1})(Y)$, analytic in the sense that for every $f \in (L^{p_0}\cap L^{p_1})(X)$ and $g \in (L^{q_0{}'}\cap L^{q_1{}'})(Y)$, the map $z \mapsto \int_Y T_z f \cdot g d\nu$ is holomorphic on S, continuous up to $\overline{S}$ and bounded on $\overline{S}$. Assume for all $f \in (L^{p_0}\cap L^{p_1})(X)$, we have

$$\|T_z f\|_{L^{q_j}} \leq A_j \|f\|_{L^{p_j}} \quad \textit{whenever } \mathrm{Re}\, z = j, \textit{ for } j = 0, 1.$$

Then for any $\theta \in (0,1)$, we have

$$\|T_\theta f\|_{L^q} \leq A_0^{1-\theta} A_1^{\theta} \|f\|_{L^p} \quad \textit{for all } f \in (L^{p_0}\cap L^{p_1})(X),$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

In particular, T_θ extends to a bounded linear map from $L^p(X)$ to $L^q(Y)$, with norm $\leq A_0^{1-\theta} A_1^{\theta}$.

- The ability to vary the operator involved makes this theorem way more powerful than the original theorem of Riesz-Thorin.
- One particularly striking aspect of this theorem is that its proof can be obtained from that of the Riesz-Thorin theorem simply "by adding a single letter of the alphabet" (i.e. by replacing T everywhere by T_z).
- Indeed, suppose $p \neq \infty$ and $q \neq 1$. By considering $F(z) = \int_Y T_z f_z \cdot g_z d\nu$ instead, we see that

$$\|T_\theta f\|_{L^q} \leq A_0^{1-\theta} A_1^\theta \|f\|_{L^p}$$

for all simple functions f on X. The continuity of $T_\theta \colon (L^{p_0} \cap L^{p_1})(X) \to (L^{q_0} + L^{q_1})(Y)$, together with the density of simple functions in $(L^{p_0} \cap L^{p_1})(X)$, shows that the same inequality is true for $f \in (L^{p_0} \cap L^{p_1})(X)$. Similarly one can adapt the previous argument if $p = \infty$ or $q = 1$.

- This proof shows that one can relax the assumption that $\int_Y T_z f \cdot g d\nu$ is bounded on S for every $f \in (L^{p_0} \cap L^{p_1})(X)$ and $g \in (L^{q_0'} \cap L^{q_1'})(Y)$, to the assumption that for every such f and g, there exist $\alpha < 1$ and constants C, c such that

$$\left| \int_Y T_z f \cdot g d\nu \right| \leq C e^{c e^{\pi \alpha |z|}} \quad \text{for all } z \in S.$$

- We remark that the above proofs of complex interpolation rely crucially on the duality between L^q and $L^{q'}$ when $q \in [1, \infty]$; this gives rise to the assumption $p_0, p_1, q_0, q_1 \in [1, \infty]$.
- But one can modify the above proof, so that the conditions on the exponents can be relaxed to $p_0, p_1, q_0, q_1 \in (0, \infty]$.
- The key is to first take appropriate 'square root' of the functions involved, and to use the maximum principle for subharmonic functions instead of that for holomorphic functions. See Homework 8 for details.
- Also see Homework 8 for a complex interpolation theorem for bilinear operators.

Complex interpolation involving BMO

- We specialize now to the case when $X = Y = \mathbb{R}^n$ with the usual Lebesgue measure.
- One can also use complex interpolation for operators that map into BMO instead of L^∞.
- Recall that a locally integrable function h on $\mathbb{R}^n$ is said to be in BMO, if the sharp maximal function $M^\sharp h$ is in L^∞, where

$$M^\sharp h(x) := \sup_{x \in B} ⨍_B |h(y) - h_B| dy,$$

  the supremum taken over all balls B containing x.
- Also recall that L^1_{loc} is the space of all locally integrable functions on $\mathbb{R}^n$, and it is a topological vector space where $f_n \to f$ in L^1_{loc}, if and only if $\|f_n - f\|_{L^1(K)} \to 0$ for every compact subset K of $\mathbb{R}^n$.
- For convenience, let us write L^∞_0 for the space of bounded, compactly supported measurable functions g on $\mathbb{R}^n$, with $\int_{\mathbb{R}^n} g dx = 0$.

Theorem

Let $p_0, p_1 \in [1,\infty]$. Suppose $\{T_z\}_{z\in\overline{S}}$ is a family of continuous linear operators from $(L^{p_0}\cap L^{p_1})(\mathbb{R}^n)$ to $L^1_{loc}(\mathbb{R}^n)$, analytic in the sense that for every simple function f and every $g \in L^\infty_0$, the map $z \mapsto \int_{\mathbb{R}^n} T_z f \cdot g dx$ is holomorphic on S, continuous up to $\overline{S}$ and bounded on $\overline{S}$. Let $q_0 \in [1,\infty)$. Assume for all $f \in L^{p_0}\cap L^{p_1}$, we have $T_z f \in L^{q_0}$ for all $z \in \overline{S}$, with

$$\begin{aligned} \|T_z f\|_{L^{q_0}} &\le A_0 \|f\|_{L^{p_0}} \quad \text{whenever } \operatorname{Re} z = 0, \\ \|T_z f\|_{BMO} &\le A_1 \|f\|_{L^{p_1}} \quad \text{whenever } \operatorname{Re} z = 1. \end{aligned}$$

Then for any $\theta \in (0,1)$, we have

$$\|T_\theta f\|_{L^q} \lesssim A_0^{1-\theta} A_1^{\theta} \|f\|_{L^p} \quad \text{for all } f \in L^{p_0}\cap L^{p_1},$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0}.$$

In particular, T_θ extends to a bounded linear map from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, with norm $\lesssim A_0^{1-\theta}A_1^{\theta}$.

- The key is the following proposition:

Proposition

Let $q_0 \in [1, \infty)$. Suppose $\{h_z\}_{z \in \overline{S}}$ is an analytic family of L^1_{loc} functions on $\mathbb{R}^n$, in the sense that for every $g \in L^\infty_0(\mathbb{R}^n)$, the map $z \mapsto \int_{\mathbb{R}^n} h_z \cdot g dx$ is holomorphic on S, continuous on $\overline{S}$ and bounded on $\overline{S}$. Assume that $h_z \in L^{q_0}$ for all $z \in \overline{S}$, and that there exists constants A_0, A_1 such that

$$\|h_z\|_{L^{q_0}} \leq A_0 \quad \text{whenever } Re\, z = 0,$$
$$\|h_z\|_{BMO} \leq A_1 \quad \text{whenever } Re\, z = 1.$$

Then for any $\theta \in (0,1)$, we have $h_\theta \in L^q$ with

$$\|h_\theta\|_{L^q} \lesssim A_0^{1-\theta} A_1^{\theta}, \quad \text{where} \quad \frac{1}{q} = \frac{1-\theta}{q_0}.$$

- Assuming the proposition for the moment, we finish the proof of the theorem as follows.

- Let $p_0, p_1, q_0 \in [1, \infty]$, $\theta \in (0, 1)$, and define p, q as in the statement of the theorem.
- Let first f be a simple function on $\mathbb{R}^n$.
- We apply our earlier proposition to f, p_0, p_1, θ, so that we have a holomorphic family f_z, with $\|f_z\|_{L^{p_j}} \leq \|f\|_{L^p}$ when $\operatorname{Re} z = j$, $j = 0, 1$, and $f_\theta = f$.
- Then $h_z := T_z f_z$ satisfies the hypothesis of the proposition on the previous slide, so for any $\theta \in (0, 1)$, we have $T_\theta f \in L^q$, with
$$\|T_\theta f\|_{L^q} \lesssim A_0^{1-\theta} A_1^{\theta} \|f\|_{L^p}.$$
- Now since simple functions are dense in $L^{p_0} \cap L^{p_1}$, if f is a general $L^{p_0} \cap L^{p_1}$ function on $\mathbb{R}^n$, we take a sequence of simple functions $\{f_n\}$ so that $f_n \to f$ in $L^{p_0} \cap L^{p_1}$.
- Then by continuity of $T_\theta \colon L^{p_0} \cap L^{p_1} \to L^1_{\text{loc}}$, we have $T_\theta f_n \to T_\theta f$ in L^1_{loc}, whereas our earlier estimate for simple functions show that $T_\theta f_n$ is Cauchy in L^q.
- Since convergence in L^q implies convergence in L^1_{loc}, this shows $T_\theta f \in L^q$, with $\|T_\theta f\|_{L^q} \lesssim A_0^{1-\theta} A_1^{\theta} \|f\|_{L^p}$ for this general $f \in L^{p_0} \cap L^{p_1}$ as well.

- This finishes the proof of the theorem.
- We now turn to the proof of the proposition. We use the following lemma:

Lemma

Suppose $h \in L^{q_0}(\mathbb{R}^n)$. *If* $M^\sharp h \in L^q(\mathbb{R}^n)$ *for some* $q \in [q_0, \infty)$, *then* $h \in L^q(\mathbb{R}^n)$ *with*

$$\|h\|_{L^q} \lesssim \|M^\sharp h\|_{L^q}.$$

- The proof of the lemma is based on a relative distributional inequality from Homework 3. See Homework 8 for details.
- In view of the lemma, to prove the proposition, we only need to show that $\|M^\sharp h_\theta\|_{L^q} \lesssim A_0^{1-\theta} A_1^\theta$ for all $\theta \in (0,1)$.
- Recall for $h \in L^1_{\text{loc}}$, $M^\sharp h(x) = \sup_{x \in B} ⨍_B |h(y) - h_B| dy$, where the supremum is taken over all balls B containing x.
- But by dominated convergence, it suffices to take balls with center in $\mathbb{Q}^n$ and radius in $\mathbb{Q}$.

- Now consider a collection of balls $\{B_x\}_{x\in\mathbb{R}^n}$ such that B_x contains x, the volume of B_x is bounded above and below independent of x, and the center and the radius of B_x depends measurably on x (such measurability could be guaranteed, if say the center and the radius takes value in a countable set like $\mathbb{Q}^n$ and $\mathbb{Q}$).
- Also consider a measurable function $\eta(x,y)$ on $\mathbb{R}^n \times \mathbb{R}^n$ with $|\eta(x,y)| \leq 1$ for all $x, y \in \mathbb{R}^n$.
- If for a fixed $x \in \mathbb{R}^n$, we compute

$$\fint_{B_x} [h(y) - h_{B_x}]\eta(x,y)dy$$

and take supremum over all collections of balls and all functions η as above, then we obtain $M^\sharp h(x)$.
- We now return to the setting of the proposition.
- We want to estimate $\|M^\sharp h_\theta\|_{L^q}$.

- By duality, we fix a compactly supported simple function g with $\|g\|_{L^{q'}} = 1$, and consider a holomorphic extension g_z of g such that g_z is a simple function for each $z \in \overline{S}$, the map $z \mapsto g_z$ is holomorphic on S, continuous on $\overline{S}$, bounded on $\overline{S}$, with

$$\|g_z\|_{L^{q_0{}'}} \leq 1 \quad \text{when } \operatorname{Re} z = 0,$$
$$\|g_z\|_{L^1} \leq 1 \quad \text{when } \operatorname{Re} z = 1.$$

- We fix any collection of balls $\{B_x\}$ and bounded function η as on the previous slide.
- Now let

$$F(z) := \int_{\mathbb{R}^n} \fint_{B_x} [h_z(y) - (h_z)_{B_x}]\eta(x,y) dy g_z(x) dx \quad z \in \overline{S}$$

where $\{h_z\}_{z \in \overline{S}}$ is as in the proposition.
- Note that if $g = \sum_j b_j \chi_{F_j}$ where the F_j's are disjoint bounded measurable subsets of $\mathbb{R}^n$, then

$$g_z(x) = \sum_j |b_j|^{q'\left(\frac{1-z}{q_0} + \frac{z}{1}\right)} \frac{b_j}{|b_j|} \chi_{F_j}.$$

- So $F(z) = \sum_j |b_j|^{q'\left(\frac{1-\theta}{q_0}+\frac{\theta}{1}\right)} \frac{b_j}{|b_j|} \int_{\mathbb{R}^n} h_z(y) G_j(y) dy$, where $G_j(y)$ is given by

$$\int_{\mathbb{R}^n} \chi_{F_j}(x) \left[\frac{\chi_{B_x}(y)}{|B_x|} \eta(x,y) - \frac{\chi_{B_x}(y)}{|B_x|} \int_{\mathbb{R}^n} \frac{\chi_{B_x}(w)}{|B_x|} \eta(x,w) dw \right] dx;$$

  note $G_j(y)$ is in L_0^∞ for every j.
- Our assumptions guarantee that F is holomorphic on S, continuous on $\overline{S}$, bounded on $\overline{S}$, and

$$|F(z)| \leq \|h_z\|_{BMO} \|g_z\|_{L^1} \leq A_1 \quad \text{when } \operatorname{Re} z = 1,$$

$$|F(z)| \leq 2 \|Mh_z\|_{L^{q_0}} \|g_z\|_{L^{q_0'}} \lesssim A_0 \quad \text{when } \operatorname{Re} z = 0.$$

- So the three lines lemma implies that

$$|F(\theta)| \lesssim A_0^{1-\theta} A_1^{\theta},$$

which in turn implies

$$\|M^{\sharp} h_{\theta}\|_{L^q} \lesssim A_0^{1-\theta} A_1^{\theta}.$$

This completes the proof of the proposition.
- We remark that the hypothesis of the proposition can be weakened as before: it will suffice if for every $g \in L_0^{\infty}$, there exist $\alpha < 1$ and C, c such that $|\int_{\mathbb{R}^n} h_z \cdot g dx| \leq Ce^{ce^{\pi\alpha|z|}}$ for all $z \in S$. This yields a corresponding improvement of the complex interpolation theorem involving BMO.

Comparing the real and complex methods of interpolation

- To conclude, let us draw a comparison between the real and complex methods of interpolation.
- The real method of interpolation allows one to convert weak-type or restricted weak-type hypothesis into strong type conclusions (whereas the complex method doesn't).
- Indeed, the real method is less sensitive to the hypothesis given at the endpoints; it gives the same conclusion regardless of whether a strong-type and a (restricted) weak-type hypothesis is given (contrary to the complex method).
- The real method also allows one to work with subadditive operators (whereas the complex method requires the operator to be linear, or at least linearizable).
- On the other hand, the complex method allows one to vary an operator within an analytic family, a feature that is tremendously useful in practice.