A positive mass theorem in 3-dimensional CR geometry

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Introduction

- Joint work with Chin-Yu Hsiao
- Solution of a tangential Kohn Laplacian $\Box_b$
- Difficulty: we will work on a non-compact CR manifold
- e.g. Even on the Heisenberg group $\mathbb{H}^n$, $\Box_b$ may not have closed range, when it is extended as a closed linear operator

$$\Box_b \colon L^2(\mathbb{H}^n) \to L^2(\mathbb{H}^n)$$

- Way out: use conformal equivalence, and extend $\Box_b$ instead as

$$\Box_b \colon L^p \to L^q$$

  (Another possibility is to consider $\Box_b$ as an operator from a weighted L^2 space to itself, as in our earlier work; we will not pursue that today)

- Application: a positive mass theorem in CR geometry, as was proposed by Cheng, Malchiodi and Yang.

Outline of the talk

- The (Riemannian) Yamabe problem
- The CR Yamabe problem
- A CR positive mass theorem in 3-dimensions
- Solution of $\Box_b$ on a certain class of non-compact 3-dimensional CR manifolds

The Yamabe problem

- (M^n, g) compact Riemannian manifold of dimension $n \geq 2$
- A metric $\hat{g}$ is said to be conformally equivalent to g, if $\hat{g} = e^{2w} g$ for some smooth function w on M.
- Question: Can one conformally change the metric g, such that the new metric $\hat{g}$ has constant scalar curvature?
- Answer: Yes.
  dimension $n = 2$: uniformization theorem
  dimension $n \geq 3$: contribution by Yamabe, Trudinger, Aubin, Scheon, Yau, ...

- In dimension $n \geq 3$, write the conformal metric as

$$\hat{g} = u^{\frac{4}{n-2}} g$$

for some positive smooth function u on M, and

$$L_g := c_n \Delta_g + R_g$$

for the conformal Laplacian. Then the Yamabe problem for (M, g) reduces to the following PDE:

$$L_g u = R_{\hat{g}} u^{\frac{n+2}{n-2}}, \quad R_{\hat{g}} = \text{constant}.$$

- This is a variational problem: it suffices to minimize the functional

$$E_g(u) := \frac{\int_M (|\nabla_g u|^2 + R_g u^2) d\mathrm{vol}_g}{\left(\int_M |u|^{\frac{2n}{n-2}} d\mathrm{vol}_g\right)^{\frac{n-2}{n}}}.$$

- Define the Yamabe constant by

$$Y(M,g) := \inf\{E_g(u) \colon u \in C^\infty(M), u > 0\}.$$

- It is known that for any compact Riemannian manifold (M^n, g), we have

$$Y(M,g) \leq Y(\mathbb{S}^n, g_{\text{std}}),$$

and the inequality is strict unless $(M,[g]) \simeq (\mathbb{S}^n, [g_{\text{std}}])$.
- If $n = 3, 4, 5$ or (M^n, g) is locally conformally flat, this last statement was established via a positive mass theorem.
- This strictness of the inequality is important, because it is known that the Yamabe problem can be resolved in the affirmative when $Y(M,g) < Y(\mathbb{S}^n, g_{\text{std}})$.
- We now discuss the analog of the Yamabe problem in 3-dimensional CR geometry.

The CR Yamabe problem: Set-up

- M: an orientable CR manifold of dimension 3, meaning that there exists a distinguished 1-dimensional subbundle L of $\mathbb{C}TM$, with $L \cap \bar{L} = \{0\}$.
- Write $\xi = \mathrm{Re}\,(L \oplus \bar{L})$.
- Assume that there exists a (real) contact form θ on M (so $\theta \wedge d\theta \neq 0$ on M), such that

$$\text{kernel}\,\theta = \xi.$$

  (In particular, this implies that M is strongly pseudoconvex.)
- Replacing θ by $-\theta$ if necessary, one can define a Hermitian inner product on L, by

$$(Z, W)_\theta := 2i d\theta(Z \wedge \bar{W}), \quad Z, W \in \Gamma(L).$$

- We call such (M, θ) a pseudohermitian manifold, and think of $\theta \wedge d\theta$ as the natural volume form on M.

- (M, θ): a pseudohermitian manifold
- Then as was first shown by Tanaka and Webster, one can define an associated connection on TM, that is compatible with the CR and pseudohermitian structures
  $\rightarrow$ define the corresponding (scalar) curvature and torsion.
- Write R_θ for the scalar curvature associated to θ.
- e.g. $(\mathbb{S}^3, \theta_{\text{std}})$: standard round sphere $\{|\zeta| = 1\}$ in $\mathbb{C}^2$,

$$L = \text{span}\left\{\overline{\zeta^2}\frac{\partial}{\partial \zeta^1} - \overline{\zeta^1}\frac{\partial}{\partial \zeta^2}\right\}, \qquad \theta_{\text{std}} := i(\bar{\partial} - \partial)|\zeta|^2.$$

  Then $R_{\theta_{\text{std}}} \equiv 1$.
- e.g. $(\mathbb{H}^1, \theta_0)$: Heisenberg group $\simeq \mathbb{C} \times \mathbb{R}$,

$$L = \text{span}\left\{\frac{\partial}{\partial z} + i\bar{z}\frac{\partial}{\partial t}\right\}, \qquad \theta_0 := dt + i(z d\bar{z} - \bar{z} dz).$$

  Then $R_{\theta_0} \equiv 0$.

Various differential operators of interest on (M, θ)

- The subgradient ∇_b:

$$\nabla_b u = (Xu, Yu)$$

where $Z := \frac{1}{2}(X + iY)$ is a local section of L with $(Z, Z)_\theta = 1$.

- The sublaplacian Δ_b:

$$\Delta_b u = (X^*X + Y^*Y)u$$

where X, Y are as above, and X^*, Y^* are their adjoint under $L^2(\theta \wedge d\theta)$.

- The Kohn Laplacian $\Box_b$:

$$\Box_b u = \overline{Z}^* \overline{Z} u$$

where $\overline{Z}$ is a local section of $\bar{L}$ with $(\overline{Z}, \overline{Z})_\theta = 1$, and $\overline{Z}^*$ is its adjoint under $L^2(\theta \wedge d\theta)$.

- The conformal sublaplacian L_b:

$$L_b f = (4\Delta_b + R_\theta)f.$$

It describes how the Tanaka-Webster scalar curvature changes under a conformal change of contact form: if $\hat{\theta} = u^2\theta$, then

$$L_b u = R_{\hat{\theta}} u^3.$$

- The CR Paneitz operator P_b:

$$P_b f = \frac{1}{4}\Box_b \overline{\Box}_b f - iZ[\mathrm{Tor}_\theta(T, \overline{Z})f]$$

where Tor_θ is the torsion of the Tanaka-Webster connection on (M, θ), $\overline{Z}$ is a local section of $\overline{L}$ with $(\overline{Z}, \overline{Z})_\theta = 1$, and T is the Reeb vector field of the contact form θ.
It can be used to describe how a certain CR Q-curvature changes under a conformal change of the contact form.

The CR Yamabe problem

- (M, θ) 3-dimensional pseudohermitian.
- If $\hat{\theta} = u^2\theta$ for some smooth function u with $u > 0$, then

$$(Z, W)_{\hat{\theta}} = u^2 (Z, W)_{\theta}, \quad Z, W \in \Gamma(L),$$

  and we say $\hat{\theta}$ is conformally equivalent to θ.
- Question: If (M, θ) is compact, can we conformally change the contact form θ, such that the new contact form $\hat{\theta}$ has Tanaka-Webster scalar curvature $R_{\hat{\theta}} =$ constant?
- This is equivalent to solving the CR Yamabe equation on M:

$$L_b u = R_{\hat{\theta}} u^3, \quad R_{\hat{\theta}} = \mathrm{constant}.$$

- The problem is again variational: it suffices to minimize the functional

$$E_\theta(u) := \frac{\int_M (|\nabla_b u|^2 + R_\theta u^2)\theta \wedge d\theta}{\left(\int_M u^4 \theta \wedge d\theta\right)^{1/2}}.$$

- Define the CR Yamabe constant by

$$Y(M,\theta) := \inf\{E_\theta(u) \colon u \in C^\infty(M), u > 0\}.$$

- It is an old result of Jerison and Lee, that for any compact 3-dimensional pseudohermitian manifolds (M,θ),

$$Y(M,\theta) \leq Y(\mathbb{S}^3, \theta_{\mathrm{std}}).$$

Also, if strict inequality holds, then $Y(M,\theta)$ is attained by a positive smooth function u on M, and the CR Yamabe problem can be resolved in the affirmative.
$\rightarrow$ Focus only on the case $Y(M,\theta) > 0$.

The Green's function of the conformal sublaplacian

- (M, θ) 3-dimensional compact pseudohermitian, $Y(M, \theta) > 0$.
- Fix a point $p \in M$.
- We study the Green's function G_p of the conformal sublaplacian of (M, θ) with pole p: in other words, G_p is singular at p, with

$$L_b G_p = 16\delta_p.$$

- Write $\rho(q)$ for a suitable non-isotropic distance from q to p.
- Also, let $\mathcal{O}^j$ be the set of all smooth functions f on $M \setminus \{p\}$, with

$$|f(q)| \lesssim \rho(q)^j,$$

and $|\nabla_b^k f(q)| \lesssim \rho(q)^{j-k}$ for $k = 1, 2, \ldots$.

- By first conformally changing the contact form on M if necessary, for $q \in M$ near p, the Green's function admits an expansion

$$G_p(q) = \frac{1}{2\pi}\rho(q)^{-2} + A + \text{error}, \quad \text{error} \in \mathcal{O}^1.$$

where A is a constant.
- This is the analog of the conformal normal coordinates in CR geometry.
- We will assume our contact form θ has been chosen already, so that the above expansion of G_p is valid near p.
- The constant A will be a positive multiple of the mass of a certain blow-up of (M, θ). Its sign will be important in the CR Yamabe problem in 3 dimensions.

A CR positive mass theorem

Theorem (Cheng-Malchiodi-Yang)

Suppose (M, θ) *is a 3-dimensional compact pseudohermitian CR manifold. Suppose in addition*

(i) $Y(M, \theta) > 0$, *and*

(ii) *the Paneitz operator* P_b *is non-negative, in the sense that* $\int_M v \cdot \overline{P_b v}\, \theta \wedge d\theta \geq 0$ *for all* $v \in C^\infty(M)$.

For any $p \in M$, *let* G_p *be the Green's function of the conformal sublaplacian* L_b *at* p, *and* A *be the constant term in the expansion of* G_p *in CR conformal normal coordinates. Then*

(a) $A \geq 0$;

(b) *If* $A = 0$ *at some point* $p \in M$, *then* M *is CR equivalent to* $\mathbb{S}^3$, *and* $[\theta] = [\theta_{std}]$.

- It follows that under the same assumptions, unless $(M, [\theta]) \simeq (\mathbb{S}^3, [\theta_{\text{std}}])$, we have $A > 0$ in the expansion of G_p.
- But when $A > 0$, one can construct a suitable test function u on M, to show that

$$E_\theta(u) < Y(\mathbb{S}^3, \theta_{\text{std}}).$$

(u is obtained by gluing G_p to a standard bubble on $(\mathbb{H}^1, \theta_0)$.)
- Hence under the assumptions of the above theorem, we have

$$Y(M, \theta) < Y(\mathbb{S}^3, \theta_{\text{std}})$$

unless $(M, [\theta]) \simeq (\mathbb{S}^3, [\theta_{\text{std}}])$, and the CR Yamabe quotient $Y(M, \theta)$ is achieved by some positive smooth minimizer.
- See also Gamara and Gamara-Jacoub, where they solved the CR Yamabe problem by seeking critical points of the functional E_θ that are not necessarily minimizers.

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- The theorem is about understanding the Green's function G_p.
- To do so, one first construct the blow-up $(M^\sharp, \theta^\sharp)$ of (M, θ), where

$$M^\sharp := M \setminus \{p\}, \quad \theta^\sharp := G_p^2 \theta.$$

- Then $(M^\sharp, \theta^\sharp)$ becomes a non-compact pseudohermitian manifold with infinite volume.
- Under a further change of coordinates, if U is a sufficiently small neighborhood of p in M, then one can identify

$$U \setminus \{p\} \subset M^\sharp \quad \leftrightarrow \quad \text{a neighborhood of infinity on } \mathbb{H}^1.$$

  Since $\mathbb{H}^1$ is flat, this allows one to identify $M^\sharp$ as an *asymptotically flat* pseudohermitian manifold.

- Example:

$$M = \mathbb{S}^3 \subset \mathbb{C}^2, \quad \theta = \theta_{\mathrm{std}} = i(\overline{\partial} - \partial)|\zeta|^2, \quad p = (0, -1)$$

- The Green's function of conformal sublaplacian on M with pole p is then $G_p = |h|$, where

$$h(\zeta_1, \zeta_2) = \frac{1}{1 + \zeta_2}.$$

- Then $(M^\sharp, \theta^\sharp) := (M \setminus \{p\}, G_p^2 \theta)$ is isometric to the Heisenberg group $(\mathbb{H}^1, \theta_0)$, where $\theta_0 = dt + i(zd\overline{z} - \overline{z}dz)$; in fact the 'stereographic projection' map

$$\zeta \in \mathbb{S}^3 \setminus \{p\} \mapsto (z, t) \in \mathbb{H}^1$$

$$z = \frac{\zeta_1}{1 + \zeta_2}, \qquad t = -\mathrm{Re}\, \frac{1 - \zeta_2}{1 + \zeta_2}$$

is an isometry between $(M^\sharp, \theta^\sharp)$ and $(\mathbb{H}^1, \theta_0)$.

- Back to our general setting, where $(M^\sharp, \theta^\sharp)$ is asymptotically flat; in particular, there exists a compact subset K of $M^\sharp$, where we identify $M^\sharp \setminus K$ with a neighborhood of infinity on $\mathbb{H}^1$.
- It turns out one can define the *mass* of such $(M^\sharp, \theta^\sharp)$, by means of an integral of certain geometric quantities on a 'sphere at infinity' on $\mathbb{H}^1$.

Proposition (Cheng-Malchiodi-Yang)

Suppose $(M^\sharp, \theta^\sharp)$ arises from the blow-up of a compact 3-dimensional pseudohermitian manifold (M, θ) as described above at some point $p \in M$. Then its mass satisfies

$$m(M^\sharp, \theta^\sharp) = 48\pi^2 A,$$

where A is the constant in the expansion of the Green's function G_p of L_b on (M, θ) at p, in CR conformal normal coordinates.

Proposition (continued)

Furthermore, there exists some function $w \in \mathcal{O}^{-1}$ on $M^\sharp$, with $\Box_b^\sharp w \in \mathcal{O}^4$, such that the mass of $(M^\sharp, \theta^\sharp)$ satisfies

$$m(M^\sharp, \theta^\sharp) = -\frac{3}{2}\int_{M^\sharp} |\Box_b^\sharp w|^2 \theta^\sharp \wedge d\theta^\sharp + 3\int_{M^\sharp} |\nabla^\sharp_{\overline{Z}^\sharp} \nabla^\sharp_{\overline{Z}^\sharp} w|^2 \theta^\sharp \wedge d\theta^\sharp + \frac{3}{4}\int_{M^\sharp} w \cdot \overline{P_b^\sharp w}\, \theta^\sharp \wedge d\theta^\sharp.$$

Here $\Box_b^\sharp$, $\nabla^\sharp$ and $P_b^\sharp$ are the Kohn Laplacian, the Tanaka-Webster connection, and CR Paneitz operator with respect to $(M^\sharp, \theta^\sharp)$, and $\overline{Z}^\sharp$ is a section of $\overline{L}$ on $M^\sharp$ with $(\overline{Z}^\sharp, \overline{Z}^\sharp)_{\theta^\sharp} = 1$.

- This is a version of Bochner's formula; one gets this by integrating by parts twice in the term involving $P_b^\sharp$.

Proposition (continued)

$$m(M^\sharp, \theta^\sharp) = -\frac{3}{2}\int_{M^\sharp} |\Box_b^\sharp w|^2 \theta^\sharp \wedge d\theta^\sharp + 3\int_{M^\sharp} |\nabla^\sharp_{\overline{Z}^\sharp}\nabla^\sharp_{\overline{Z}^\sharp} w|^2 \theta^\sharp \wedge d\theta^\sharp$$
$$+ \frac{3}{4}\int_{M^\sharp} w \cdot \overline{P_b^\sharp w}\, \theta^\sharp \wedge d\theta^\sharp.$$

In addition, the same continues to hold, when w is replaced by any v on $M^\sharp$, with $v - w \in \mathcal{O}^{1+\delta}$ and $\Box_b^\sharp v \in \mathcal{O}^{3+\delta}$ for some $\delta > 0$.

Theorem (Hsiao-Y.)

Under the assumptions of the 3-dim CR positive mass theorem, namely that $Y(M, \theta) > 0$ and $P_b \geq 0$ on (M, θ), there exists a smooth function v on $M^\sharp$, such that

$$v - w \in \mathcal{O}^{1+\delta} \quad \textit{for all } \delta \in (0, 1), \textit{ and} \quad \Box_b^\sharp v = 0.$$

- As a result, the formula for mass simplifies:

$$m(M^\sharp,\theta^\sharp) = 3\int_{M^\sharp} |\nabla^\sharp_{\overline{Z}^\sharp}\nabla^\sharp_{\overline{Z}^\sharp} v|^2 \theta^\sharp \wedge d\theta^\sharp + \frac{3}{4}\int_{M^\sharp} v \cdot \overline{P^\sharp_b v}\, \theta^\sharp \wedge d\theta^\sharp.$$

  With a little more work to bring the integral involving $P^\sharp_b$ under control, we can show that $m(M^\sharp,\theta^\sharp) \geq 0$.
  (In fact the integral involving $P^\sharp_b$ can be written as the sum of a non-negative term with $-\frac{4}{3}m(M^\sharp,\theta^\sharp)$, the latter of which can be reabsorbed into the left hand side.)
- Recalling the relation between $m(M^\sharp,\theta^\sharp)$ and the constant term A in the expansion of the Green's function G_p at p, one sees that

$$A = \frac{1}{48\pi^2} m(M^\sharp,\theta^\sharp) \geq 0.$$

- Further work then allows one to characterize when A is zero at some point p.

Solving $\Box_b^\sharp$

- Recall the statement of our theorem: $w \in \mathcal{O}^{-1}$ is a given function on M, with $\Box_b^\sharp w \in \mathcal{O}^4$.

Theorem (Hsiao-Y.)

If $Y(M,\theta) > 0$ and $P_b \geq 0$ on (M,θ), then there exists a smooth function v on $M^\sharp$, such that

$$v - w \in \mathcal{O}^{1+\delta} \quad \text{for all } \delta \in (0,1), \text{ and } \quad \Box_b^\sharp v = 0.$$

- To prove this, let $f = \Box_b^\sharp w \in \mathcal{O}^{3+\delta}$ for all $\delta \in (0,1)$.
- We solve $\Box_b^\sharp u = f$ for $u \in \mathcal{O}^{1+\delta}$ with estimates.
- Hence taking $v = w - u$, we have all conclusions of our theorem, namely $v - w \in \mathcal{O}^{1+\delta}$, and $\Box_b^\sharp v = 0$.
- Thus the key is to solve the Kohn Laplacian on $(M^\sharp, \theta^\sharp)$. This is done via the conformal equivalence between $\theta^\sharp$ with θ.

A toy problem

- We saw how $(\mathbb{H}^1, \theta_0)$ arises as the blow-up of $(\mathbb{S}^3, \theta_{\text{std}})$.
- We know very well how one could solve the Kohn Laplacian $\Box_b$ on $(\mathbb{S}^3, \theta_{\text{std}})$.
- Question: Can we use this knowledge to solve

$$\Box_b^\sharp u = f \quad \text{on } (\mathbb{H}^1, \theta_0)?$$

- The key here turns out to be that not only $\theta_0 = G_p^2 \theta_{\text{std}}$, but also there exists a CR function h on $\mathbb{S}^3 \setminus \{p\}$, i.e. one with

$$\overline{Z} h = 0, \quad \text{such that} \quad G_p = |h|.$$

  In fact, as we saw before, in this case one can take h to be

$$h(\zeta_1, \zeta_2) = \frac{1}{1 + \zeta_2}.$$

- Let $\overline{Z}$ be a section of $\bar{L}$ on $\mathbb{S}^3$ with $(\overline{Z}, \overline{Z})_{\theta_{\text{std}}} = 1$.
- Write $\overline{Z}^*$ for its formal adjoint under $L^2(\mathbb{S}^3, \theta_{\text{std}} \wedge d\theta_{\text{std}})$.
- Then $\overline{Z}^\sharp := h^{-1}\overline{Z}$ is a section of $\bar{L}$ on $\mathbb{H}^1$, with $(\overline{Z}^\sharp, \overline{Z}^\sharp)_{\theta_0} = 1$.
- Also, the formal adjoint of $\overline{Z}^\sharp$ under $L^2(\mathbb{H}^1, \theta_0 \wedge d\theta_0)$ is given by

  $$(\overline{Z}^\sharp)^* v = |h|^{-4}\overline{Z}^*(h|h|^2 v);$$

  this follows since $\theta_0 \wedge d\theta_0 = |h|^4 \theta_{\text{std}} \wedge d\theta_{\text{std}}$. In fact,

  $$\begin{aligned} \int \overline{Z}^\sharp u \cdot \overline{v}\, \theta_0 \wedge d\theta_0 &= \int h^{-1}\overline{Z} u \cdot \overline{v} |h|^4\, \theta_{\text{std}} \wedge d\theta_{\text{std}} \\ &= \int u \cdot \overline{\overline{Z}^*(h|h|^2 v)}\, \theta_{\text{std}} \wedge d\theta_{\text{std}} \\ &= \int u \cdot \overline{|h|^{-4}\overline{Z}^*(h|h|^2 v)}\, \theta_0 \wedge d\theta_0. \end{aligned}$$

$$\overline{Z}^{\sharp} u = h^{-1}\overline{Z} u, \qquad (\overline{Z}^{\sharp})^* v = |h|^{-4}\overline{Z}^*(h|h|^2 v), \qquad \Box_b^{\sharp} = (\overline{Z}^{\sharp})^*\overline{Z}^{\sharp}.$$

- Hence

$$\Box_b^{\sharp} u = |h|^{-4}\overline{Z}^*(h|h|^2 \cdot h^{-1}\overline{Z} u) = |h|^{-4}\bar{h}\overline{Z}^*\overline{Z}(hu),$$

the last equality following from the commutativity about $\overline{Z}$ and h. In other words,

$$\Box_b^{\sharp} u = \bar{h}^{-1} h^{-2}\Box_b(hu).$$

- Thus to solve $\Box_b^{\sharp} u = f$ on $\mathbb{H}^1$, one could solve instead

$$\Box_b(hu) = \bar{h}h^2 f \quad \text{on } \mathbb{S}^3;$$

one can do this using standard theory about solutions of $\Box_b$.

The general case

- Back to the general case, where $M^\sharp = M \setminus \{p\}$, and $\theta^\sharp = G_p^2\theta$. Then it is not necessarily true that

$$G_p = |h|$$

  for some CR function h.
- Good news: one can still construct a CR function h, so that

$$|h|^2 G_p^{-2} = 1 + a, \qquad \text{for some error } a \in \mathcal{O}^2.$$

- Bad news: The error a may not be smooth across p.

A tale of 3 different $\Box_b$'s

- Goal: to solve $\Box_b^\sharp$ on $M^\sharp$
- Step 1: Introduce $\tilde{\Box}_b$ on M, such that $\Box_b^\sharp$ is conjugate to $\tilde{\Box}_b$.
- Problem: $\tilde{\Box}_b$ will in general have non-smooth coefficients
- Way out: Construct $\hat{\Box}_b$, with smooth coefficients, that approximates $\tilde{\Box}_b$

- Let $\overline{Z}$ be a local section of $\bar{L}$ on M, with $(\overline{Z}, \overline{Z})_\theta = 1$.
- Let $\overline{Z}^\sharp := G_p^{-1}\overline{Z}$, and define its Hilbert space closure

$$\overline{Z}^\sharp \colon L^2(\theta^\sharp \wedge d\theta^\sharp) \to L^2(\theta^\sharp \wedge d\theta^\sharp).$$

Let $(\overline{Z}^\sharp)^*$ be its adjoint. Then

$$\Box_b^\sharp = (\overline{Z}^\sharp)^* \overline{Z}^\sharp.$$

- Define two (possibly non-smooth) measures

$$\tilde{m}_0 = (1+\chi a)^{-1}\theta \wedge d\theta, \qquad \tilde{m}_1 = G_p^2 |h|^{-2} \theta \wedge d\theta.$$

  Here χ is a smooth function, which is identically 1 near p, and vanishes outside a small neighborhood of p.

- $\tilde{m}_0$ and $\tilde{m}_1$ are finite measures on M, which we think of as perturbations of $\theta \wedge d\theta$. In fact

$$\tilde{m}_0 = \tilde{m}_1 = \theta \wedge d\theta \quad \text{when} \quad a = 0.$$

- Let $\tilde{\overline{Z}} := G_p \overline{Z}^\sharp$, and define its Hilbert space closure

$$\tilde{\overline{Z}} \colon L^2(\tilde{m}_0) \to L^2(\tilde{m}_1).$$

  Let $\tilde{\overline{Z}}^*$ be its adjoint. Define

$$\tilde{\Box}_b := \tilde{\overline{Z}}^* \tilde{\overline{Z}}.$$

- One can check that for any function u,

$$\Box_b^\sharp u = (1+\chi a)^{-1} G_p^{-4} \overline{h} \tilde{\Box}_b (h^{-1} u).$$

- Hence solving $\Box_b^\sharp u = f$ is the same as solving

$$\tilde{\Box}_b (h^{-1} u) = (1+\chi a) G_p^4 \overline{h}^{-1} f.$$

- Problem: $\tilde{\Box}_b$ is defined using two possibly non-smooth measures $\tilde{m}_0$ and $\tilde{m}_1$. The standard theory of Kohn Laplacians do not cover this!
- The way out: construct a smooth Kohn Laplacian $\hat{\Box}_b$, which approximates $\tilde{\Box}_b$.

- Define two new measures

$$\hat{m}_0 = \theta \wedge d\theta, \qquad \hat{m}_1 = (1 + \chi a) G_p^2 |h|^{-2} \theta \wedge d\theta.$$

so that near p,

$$\hat{m}_1 = (1 + a) G_p^2 |h|^{-2} \theta \wedge d\theta = \theta \wedge d\theta.$$

- In particular, $\hat{m}_0$ and $\hat{m}_1$ are both smooth across p.
- Let $\overline{Z}$ be as before with $(\overline{Z}, \overline{Z})_\theta = 1$, and $\hat{\overline{Z}} := \overline{Z}$.

  We extend $\hat{\overline{Z}}$ to its Hilbert space closure

$$\hat{\overline{Z}} \colon L^2(m_0) \to L^2(m_1).$$

  Let $\hat{\overline{Z}}^*$ be its adjoint. Define

$$\hat{\Box}_b := \hat{\overline{Z}}^* \hat{\overline{Z}}.$$

- $\hat{\Box}_b$ is not quite the standard Kohn Laplacian $\Box_b$ on M, since the adjoint $\hat{\overline{Z}}^*$ is taken with respect to two different measures; but the standard theory of Kohn Laplacians carry over easily.
- By a result of Chanillo-Chiu-Yang, the conditions $Y(M,\theta) > 0$ and $P_b \geq 0$ implies that

$$\hat{\Box}_b \colon L^2(\theta \wedge d\theta) \to L^2(\theta \wedge d\theta) \quad \text{has closed range.}$$

  So we know in principle how to solve $\hat{\Box}_b$.
- But one can check that there exists a function $g \in \mathcal{O}^1$, with a sufficiently small support near p, such that

$$\tilde{\Box}_b = \hat{\Box}_b + g\overline{Z}.$$

- One can then solve $\tilde{\Box}_b$ using the solution operator for $\hat{\Box}_b$, by adding up a suitable Neumann series. The key is the estimates of various solution operators in $L^p(\theta \wedge d\theta)$ and $\mathcal{O}^\alpha$.