A PROOF OF THE SIMPLICITY OF $\mathfrak{sl}_n$

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Let k be a field with $\operatorname{char} k = 0$. Let $\mathfrak{sl}_n$ be the Lie subalgebra of $\mathfrak{gl}_n$ over k, defined by

$$\mathfrak{sl}_n = \{x \in \mathfrak{gl}_n \colon \operatorname{tr} x = 0\}.$$

In this note, we present a less computational proof of the following theorem:

Theorem 1. *$\mathfrak{sl}_n$ is simple.*

Let E_{ij} be the elementary matrix whose entry in the i-th row, j-th column is 1, and zero in all other entries. The key here is to show that any non-zero ideal of $\mathfrak{sl}_n$ contains at least one E_{ij} for some $i \neq j$. There are various ways of achieving this, some more elementary, at the expense of being a little more computational; the argument we present below is more conceptual, and is a variant of one given by Crystal Hoyt in a very nice set of lecture notes on Lie Algebras. The key is the following lemma, which is interesting in its own right:

Lemma 1. *Suppose V is a finite dimensional vector space, and $T \colon V \to V$ is a diagonalizable linear map. We write Λ for the set of eigenvalues of T, and $V_\lambda = \{v \in V \colon Tv = \lambda v\}$ be the eigenspace of T associated with λ, so that $V = \oplus_{\lambda \in \Lambda} V_\lambda$. If W is a T-invariant subspace of V, i.e. if $T(W) \subseteq W$, then*

$$W = \oplus_{\lambda \in \Lambda} (W \cap V_\lambda).$$

Proof. Since $V = \oplus_{\lambda \in \Lambda} V_\lambda$, given $w \in W$, one can write

$$w = \sum_{\lambda \in \Lambda} w_\lambda, \quad \text{with } w_\lambda \in V_\lambda.$$

We enumerate those $\lambda \in \Lambda$ for which $w_\lambda \neq 0$ by $\lambda_1, \dots, \lambda_m$. Then

$$\begin{aligned} w &= w_{\lambda_1} + \dots + w_{\lambda_m} \\ Tw &= \lambda_1 w_{\lambda_1} + \dots + \lambda_m w_{\lambda_m} \\ &\vdots \\ T^{m-1} w &= \lambda_1^{m-1} w_{\lambda_1} + \dots + \lambda_m^{m-1} w_{\lambda_m} \end{aligned}$$

Now we consider the coefficient matrix on the right hand side. Its determinant is the Vandermonde determinant, which is non-zero in our case since the λ_i's are all distinct:

$$\det \begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_m \\ \vdots & & \ddots & \vdots \\ \lambda_1^{m-1} & \lambda_2^{m-1} & \dots & \lambda_m^{m-1} \end{pmatrix} = \prod_{i > j} (\lambda_i - \lambda_j) \neq 0.$$

Thus we can invert this coefficient matrix. As a result, $w_{\lambda_1}, \dots, w_{\lambda_m}$ can all be written as linear combinations of $w, Tw, \dots, T^{m-1}w$, all of which are in W by the T-invariance of W. It follows that $w \in \oplus_{\lambda \in \Lambda} (W \cap V_\lambda)$, and the lemma follows. $\square$

We remark that the above lemma works for vector spaces over any field. Moreover, while we will not need this below, there is a version of this lemma involving r commuting diagonalizable linear maps $T_1, \dots, T_r \colon V \to V$, and a subspace W of V that is preserved by each of the $T_1, \dots, T_r$.

We are now ready to prove the simplicity of $\mathfrak{sl}_n$.

Proof of Theorem 1. Since $\mathfrak{gl}_n = \mathfrak{sl}_n \oplus (k \cdot I)$ where $k \cdot I$ is the set of all scalar multiples of the identity matrix I, and since $k \cdot I$ is a subset of the center of $\mathfrak{gl}_n$, if I is any ideal of $\mathfrak{sl}_n$, it is also an ideal of $\mathfrak{gl}_n$. Hence to prove Theorem 1, it suffices to prove that every non-zero ideal I of $\mathfrak{gl}_n$ with $I \subseteq \mathfrak{sl}_n$ is equal to $\mathfrak{sl}_n$.

Suppose now I is a non-zero ideal of $\mathfrak{gl}_n$ with $I \subseteq \mathfrak{sl}_n$. Let $s = \sum_{k=1}^n 2^k E_{kk}$. Then $\operatorname{ad} s$ is diagonalizable on $V := \mathfrak{gl}_n$: in fact

$$\operatorname{ad} s(E_{ij}) = (2^i - 2^j) E_{ij} \quad \text{for all } i, j = 1, \dots, n.$$

Hence the distinct eigenvalues of $\operatorname{ad} s$ are 0 and $\pm(2^i - 2^j)$, $1 \leq j < i \leq n$. (Note that if $i \neq j$ and $i' \neq j'$, then $2^i - 2^j \neq 2^{i'} - 2^{j'}$, unless $i = i'$ and $j = j'$. This is the place where we use our assumption that our base field k has characteristic zero.) Let V_λ be the eigenspace of $\operatorname{ad} s \colon V \to V$ with eigenvalue λ. Then V_0 is the set of all diagonal matrices in $\mathfrak{gl}_n$, and $V_{2^i - 2^j} = \operatorname{span}\{E_{ij}\}$ for $i \neq j$. Now I is $\operatorname{ad} s$-invariant, since I is an ideal of $\mathfrak{gl}_n$. By the previous lemma, I is the direct sum of $I \cap V_0$ with $\oplus_{i \neq j} (I \cap V_{2^i - 2^j})$. If $I \cap V_0 \neq \{0\}$, then since $I \subseteq \mathfrak{sl}_n$, one sees that I contains a matrix t of the form $\sum_{k=1}^n \lambda_k E_{kk}$, with $\lambda_i \neq \lambda_j$ for some $i \neq j$. But then I contains $[t, E_{ij}] = (\lambda_i - \lambda_j) E_{ij}$, so I contains E_{ij} for some $i \neq j$. On the other hand, if $I \cap V_0 = \{0\}$, then since I is non-zero, we must have $I \cap V_{2^i - 2^j} \neq \{0\}$ for some $i \neq j$. Hence I contains E_{ij} for some $i \neq j$ in either case.

To proceed further, it suffices to note that then I contains E_{ik} whenever $k \neq i$. and E_{kj} whenever $k \neq j$. This is because

$$[E_{jk}, E_{ij}] = -E_{ik} \quad \text{if } k \neq i,$$

and

$$[E_{ki}, E_{ij}] = E_{kj} \quad \text{if } k \neq j,$$

Repeating this argument, I contains E_{kl} for all $k \neq l$. It follows that I contains $E_{kk} - E_{ll}$ for all $k \neq l$, since

$$[E_{kl}, E_{lk}] = E_{kk} - E_{ll}.$$

One then concludes that I contains $\mathfrak{sl}_n$. Since we assumed $I \subseteq \mathfrak{sl}_n$, we obtain $I = \mathfrak{sl}_n$ as desired. This completes our proof. □