CONSEQUENCES OF THE REVERSED SQUARE FUNCTION ESTIMATE FOR THE PARABOLOID IN $\mathbb{R}^{n+1}$

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Fix $n \geq 1$. Following Carbery [1], we explain how a reversed square function estimate for the paraboloid in $\mathbb{R}^{n+1}$ implies a Kakeya estimate in $\mathbb{R}^{n+1}$, and a local smoothing estimate for the Schrödinger equation in $\mathbb{R}^{n+1}$.

Notations. *For $R \gg 1$, let $\mathcal{P}_R$ be the covering of the unit ball B_1 in the frequency space $\mathbb{R}^n$ by squares of side lengths $2R^{-1}$ with centers at $R^{-1}\mathbb{Z}^n \cap [-1,1]^n$. For $\theta \in \mathcal{P}_R$, let $\mathfrak{R}_\theta$ be a truncated neighborhood of the paraboloid in $\mathbb{R}^{n+1}$ given by*

$$\mathfrak{R}_\theta := \{(\xi, |\xi|^2 + \tau) \in \mathbb{R}^{n+1} \colon \xi \in \theta, |\tau| \leq R^{-2}\}.$$

Definition. *For $2 \leq p \leq \infty$ and $\sigma \geq 0$, we denote by $RS(p,\sigma)$ the following statement: For any $R \gg 1$, and any family of functions $\{F_\theta\}_{\theta \in \mathcal{P}_R}$ on $\mathbb{R}^{n+1}$ with support of $\widehat{F_\theta}$ contained in $\mathfrak{R}_\theta$ for every $\theta \in \mathcal{P}_R$, we have*

$$\Big\| \sum_{\theta \in \mathcal{P}_R} F_\theta \Big\|_{L^p(\mathbb{R}^{n+1})} \lesssim_{p,\sigma} R^\sigma \Big\| \Big(\sum_{\theta \in \mathcal{P}_R} |F_\theta|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^{n+1})}. \tag{1}$$

Definition. *For $2 \leq p \leq \infty$ and $s \geq 0$, we denote by $LS(p,s)$ the following statement: For any $R \gg 1$ and any Schwartz function g on $\mathbb{R}^n$ whose Fourier transform is supported on the annulus $\{R \leq |\xi| \leq 2R\}$, we have*

$$\|e^{-\frac{it\Delta}{2\pi}} g\|_{L^p(\mathbb{R}^n \times [0,1])} \lesssim_{p,s} R^s \|g\|_{L^p(\mathbb{R}^n)}. \tag{2}$$

Definition. *For $1 \leq q \leq \infty$ and $\kappa \geq 0$, we denote by $K(q,\kappa)$ the following statement: For any $R \gg 1$, and any family of cylinders $\mathbb{T}$ in $\mathbb{R}^{n+1}$ of dimensions $R^{-1} \times \cdots \times R^{-1} \times 1$ that point in R^{-1} separated directions, we have*

$$\Big\| \sum_{T \in \mathbb{T}} a_T \mathbf{1}_T \Big\|_{L^q(\mathbb{R}^{n+1})} \lesssim_{q,\kappa} R^\kappa \Big(\sum_{T \in \mathbb{T}} a_T^q |T| \Big)^{1/q} \tag{3}$$

where $\{a_T\}_{T \in \mathbb{T}}$ is any collection of non-negative real numbers indexed by $\mathbb{T}$.

Let

$$\sigma(p) := \max\left\{0, \left[n\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{p}\right]\right\} = \max\left\{0, \frac{n}{2} - \frac{n+1}{p}\right\},$$

$$s(p) := \max\left\{0, 2\left[n\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{p}\right]\right\} = \max\left\{0, n - \frac{2(n+1)}{p}\right\} \quad \text{and}$$

$$\kappa(q) := \max\left\{0, n\left(1 - \frac{1}{q}\right) - \frac{1}{q}\right\} = \max\left\{0, n - \frac{n+1}{q}\right\};$$

note that

$$s(p) = 2\sigma(p) = \kappa(p/2).$$

In dimension $n = 1$, it is known that the reversed square function estimate $RS(p, \sigma)$ holds on $\mathbb{R}^{n+1}$ for all $2 \le p \le \infty$ and all $\sigma \ge \sigma(p)$, the local smoothing estimate $LS(p, s)$ holds on $\mathbb{R}^{n+1}$ for all $2 \le p \le \infty$ and all $s > s(p)$, and the Kakeya maximal estimate $K(q, \kappa)$ holds on $\mathbb{R}^{n+1}$ for all $1 \le q \le \infty$ and all $\kappa > \kappa(q)$. In dimensions $n > 1$, it is conjectured that $RS(p, \sigma)$ holds for all $2 \le p \le \infty$ and all $\sigma > \sigma(p)$, $LS(p, s)$ holds for all $2 \le p \le \infty$ and all $s > s(p)$, and that $K(q, \kappa)$ holds for all $1 \le q \le \infty$ and all $\kappa > \kappa(q)$; none of them is known in full, despite numerous partial results.

Below we prove the following theorems.

Theorem 1. *Let $2 \le p \le \infty$ and $\sigma \ge 0$. Then $RS(p, \sigma)$ implies $K(\frac{p}{2}, 4\sigma + \max\{\frac{4(n+1)}{p} - 2n, 0\} + \varepsilon)$ for any $\varepsilon > 0$, and if $p \neq \frac{2(n+1)}{n}$, this holds for $\varepsilon = 0$ as well.*

Theorem 2. *Let $2 \le p \le \infty$, $\sigma \ge 0$ and $\kappa \ge 0$. Then $RS(p, \sigma)$ and $K(\frac{p}{2}, \kappa)$ together implies $LS(p, \sigma + \frac{\kappa}{2})$.*

Combining Theorems 1 and 2, we see that if p_c is the critical exponent $\frac{2(n+1)}{n}$, then

$$\text{“}RS(p_c, \sigma) \text{ is true for all } \sigma > 0\text{”} \Rightarrow \text{“}LS(p_c, s) \text{ is true for all } s > 0\text{”},$$

which implies the full local smoothing conjecture for all $2 \le p \le \infty$ by interpolating against the trivial L^2 and L^∞ bounds.

The proof of Theorem 1 relies on the following simple Kakeya bound, for cylinders with a common center.

Lemma 1. *Let $1 \le q \le \infty$, $R \gg 1$ and $\mathbb{T}$ be a family of cylinders in $\mathbb{R}^{n+1}$, of dimensions $R^{-1} \times \cdots \times R^{-1} \times 1$, that point in R^{-1} separated directions. If all $T \in \mathbb{T}$ are centered at the origin, then for any non-negative coefficients $\{a_T\}_{T \in \mathbb{T}}$, we have*

$$\Big\| \sum_{T \in \mathbb{T}} a_T \mathbf{1}_T \Big\|_{L^q(\mathbb{R}^{n+1})} \lesssim_q (\log R)^{e(q)} R^{\max\{0, n - \frac{n+1}{q}\}} \Big(\sum_{T \in \mathbb{T}} a_T^q |T| \Big)^{1/q}$$

where

$$e(q) := \begin{cases} \frac{1}{q} & \text{if } q = \frac{n+1}{n} \\ 0 & \text{if } q \neq \frac{n+1}{n}. \end{cases}$$

Proof. We decompose the unit ball in $\mathbb{R}^{n+1}$ into the union of the ball $B(0, R^{-1})$, centered at the origin and of radius R^{-1}, and the annuli A_k, over $k = 1, \dots, \log_2 R$, where $A_k = \{(x, t) \in \mathbb{R}^{n+1} \colon 2^{-k} \le |(x, t)| \le 2^{-(k-1)}\}$. First,

$$\int_{B(0,R^{-1})} \Big(\sum_{T \in \mathbb{T}} a_T \mathbf{1}_T \Big)^q \le \Big(\sum_{T \in \mathbb{T}} a_T \Big)^q |B(0, R^{-1})| \simeq \Big(\sum_{T \in \mathbb{T}} a_T |T| \Big)^q R^{nq-(n+1)} \lesssim \sum_{T \in \mathbb{T}} a_T^{\frac{n+1}{n}} |T| R^{nq-(n+1)}$$

by Hölder's inequality. Next, for $k = 1, \dots, \log_2 R$, we choose m_k many cylinders $T_1, T_2, \dots, T_{m_k} \in \mathbb{T}$ with $m_k \lesssim 2^{-kn} R^n$ so that $T_1, \dots, T_{m_k}$ covers the intersection of A_k with the support of $\sum_{T \in \mathbb{T}} a_T \mathbf{1}_T$, and every $T \in \mathbb{T}$ satisfies $\angle(T, T_i) \lesssim 2^k R^{-1}$ for only $O(1)$ many i's. Then by Hölder's inequality,

$$\begin{aligned}\int_{A_k} \Big(\sum_{T \in \mathbb{T}} a_T \mathbf{1}_T \Big)^q &\le \sum_{i=1}^{m_k} \Big(\sum_{T \in \mathbb{T},\, \angle(T,T_i) \lesssim 2^k R^{-1}} a_T \Big)^q |A_k \cap T_i| \\ &\lesssim \sum_{i=1}^{m_k} \Big(\sum_{T \in \mathbb{T},\, \angle(T,T_i) \lesssim 2^k R^{-1}} a_T^q \Big) (2^{kn})^{q-1} 2^{-k} |T_i| \\ &\lesssim 2^{k(nq-(n+1))} \sum_{T \in \mathbb{T}} a_T^q |T|.\end{aligned}$$

It follows that

$$\int_{\mathbb{R}^{n+1}} \Big(\sum_{T \in \mathbb{T}} a_T \mathbf{1}_T \Big)^q \lesssim \Big(R^{nq-(n+1)} + \sum_{k=1}^{\log_2 R} 2^{k(nq-(n+1))} \Big) \sum_{T \in \mathbb{T}} a_T^q |T|.$$

Our desired conclusion then follows from the estimates

$$R^{nq-(n+1)} + \sum_{k=1}^{\log_2 R} 2^{k(nq-(n+1))} \lesssim_q \begin{cases} R^{nq-(n+1)} & \text{if } \frac{n+1}{n} < q \le \infty \\ \log R & \text{if } q = \frac{n+1}{n} \\ 1 & \text{if } 1 \le q < \frac{n+1}{n} \end{cases}.$$

□

We will apply Lemma 1 to bound $\left\| \sum_{T \in \mathbb{T}} a_T \mathbf{1}_T \right\|_{L^q(\mathbb{R}^{n+1})}$ for $\mathbb{T} = \{T_\theta \colon \theta \in \mathcal{P}_R\}$, where

$$T_\theta := \{(x,t) \in \mathbb{R}^{n+1} \colon |x + 2tc_\theta| \le R^{-1}, |t| \le 1\} \tag{4}$$

for $\theta \in \mathcal{P}_R$; here c_θ denotes the center of the square θ. Indeed, we need the following slightly more general estimate, which is what we actually need in proving Theorem 1.

Lemma 2. *Let $1 \le q \le \infty$ and $R \gg 1$. For $\theta \in \mathcal{P}_R$, define T_θ by (4) where c_θ is the center of the square θ. Then for any non-negative coefficients $\{a_\theta\}_{\theta \in \mathcal{P}_R}$ and any $N > n+1$, we have*

$$\Big\| \sum_{\theta \in \mathcal{P}_R} a_\theta (1 + R|x + 2tc_\theta| + |t|)^{-N} \Big\|_{L^q(\mathbb{R}^{n+1})} \lesssim_q (\log R)^{e(q)} R^{\max\{0, n - \frac{n+1}{q}\}} \Big(\sum_{\theta \in \mathcal{P}_R} a_\theta^q |T_\theta| \Big)^{1/q}.$$

Proof of Lemma 2. Note that by Minkowski inequality,

$$\Big\| \sum_{\theta \in \mathcal{P}_R} a_\theta (1 + R|x + 2tc_\theta| + |t|)^{-N} \Big\|_{L^q(\mathbb{R}^{n+1})} \lesssim \sum_{m \in \mathbb{Z}^{n+1}} (1 + |m|)^{-N} \Big\| \sum_{\theta \in \mathcal{P}_R} a_\theta \mathbf{1}_{T_{\theta,m}} \Big\|_{L^q(\mathbb{R}^{n+1})}$$

where $T_{\theta,m} := T_\theta + m'(R^{-1}, 0) + m''(-2c_\theta, 1)$ for $m = (m', m'') \in \mathbb{Z}^n \times \mathbb{Z}$. The above L^q norm is independent of m' by translation invariance. For $|m''| \ge 1$, the cylinders $T_{\theta,(0,m'')}$ are disjoint, so

$$\Big\| \sum_{\theta \in \mathcal{P}_R} a_\theta \mathbf{1}_{T_{\theta,m}} \Big\|_{L^q(\mathbb{R}^{n+1})} = \Big(\sum_{\theta \in \mathcal{P}_R} a_\theta^q |T_\theta| \Big)^{1/q}.$$

Since $N > n+1$, it follows that

$$\Big\| \sum_{\theta \in \mathcal{P}_R} a_\theta (1 + R|x + 2tc_\theta| + |t|)^{-N} \Big\|_{L^q(\mathbb{R}^{n+1})} \lesssim \Big(\sum_{\theta \in \mathcal{P}_R} a_\theta^q |T_\theta| \Big)^{1/q} + \Big\| \sum_{\theta \in \mathcal{P}_R} a_\theta \mathbf{1}_{T_\theta} \Big\|_{L^q(\mathbb{R}^{n+1})}$$
$$\lesssim_q (\log R)^{e(q)} R^{\max\{0, n - \frac{n+1}{q}\}} \Big(\sum_{\theta \in \mathcal{P}_R} a_\theta^q |T_\theta| \Big)^{1/q}$$

where we invoked Lemma 1 in the last inequality. □

The following wave packet computation will be useful both for proving Theorems 1 and 2.

Lemma 3. *Let Φ be a Schwartz function on $\mathbb{R}^n$ whose Fourier transform is compactly supported on $[-1,1]^n$. For $R \gg 1$ and $\theta \in \mathcal{P}_R$, let Φ_θ be given by*

$$\Phi_\theta(x) := \Phi(R^{-1}x) e^{2\pi i x \cdot c_\theta}$$

where c_θ is the center of the square θ. Then

$$e^{-\frac{it\Delta}{2\pi}} \Phi_\theta(x) = e^{2\pi i (x \cdot c_\theta + t|c_\theta|^2)} \int_{\mathbb{R}^n} \widehat{\Phi}(\xi) e^{2\pi i R^{-1}(x + 2tc_\theta)\cdot \xi} e^{2\pi i R^{-2} t |\xi|^2} d\xi. \tag{5}$$

In particular,

$$e^{-\frac{it\Delta}{2\pi}} \Phi_\theta(x) = e^{2\pi i (x \cdot c_\theta + t|c_\theta|^2)} \Phi(R^{-1}(x + 2tc_\theta)) + O(R^{-2}t), \tag{6}$$

and if $\eta(t)$ is a Schwartz function on $\mathbb{R}$, then

$$|\eta(R^{-2}t) e^{-\frac{it\Delta}{2\pi}} \Phi_\theta(x)| \lesssim_N (1 + R^{-1}|x + 2tc_\theta| + R^{-2}|t|)^{-N} \tag{7}$$

for every positive integer N.

Proof. Note that

$$e^{-\frac{it\Delta}{2\pi}} \Phi_\theta(x) = R^n \int_{\mathbb{R}^n} \widehat{\Phi}(R(\xi - c_\theta)) e^{2\pi i (x\cdot\xi + t|\xi|^2)} d\xi.$$

We Taylor expand the phase $t|\xi|^2 + x \cdot \xi$ around $\xi = c_\theta$, and obtain

$$e^{-\frac{it\Delta}{2\pi}} \Phi_\theta(x) = e^{2\pi i (x \cdot c_\theta + t|c_\theta|^2)} R^n \int_{\mathbb{R}^n} \widehat{\Phi}(R(\xi - c_\theta)) e^{2\pi i (x + 2tc_\theta)\cdot(\xi - c_\theta)} e^{2\pi i t |\xi - c_\theta|^2} d\xi$$

which gives (5). We then Taylor expand the last exponential in (5) via

$$e^{2\pi i R^{-2} t |\xi|^2} = 1 + O(R^{-2}|t||\xi|^2)$$

and that gives (6). Since

$$e^{2\pi i R^{-1}(x + 2tc_\theta)\cdot\xi} = \frac{\Delta_\xi e^{2\pi i R^{-1}(x+2tc_\theta)\cdot\xi}}{(2\pi i R^{-1}|x + 2tc_\theta|)^2},$$

we may integrate by parts on the right hand side of (5), and obtain, for every positive integer N, that

$$\left| e^{-\frac{it\Delta}{2\pi}} \Phi_\theta(x) \right| \lesssim_N (1 + R^{-1}(1 + R^{-2}t)^{-1}|x + 2tc_\theta|)^{-N}.$$

Together with the rapid decay of η at infinity, we obtain the upper bound (7). □

Proof of Theorem 1. Suppose $2 \leq p \leq \infty$, $\sigma \geq 0$ and $RS(p,\sigma)$ holds. We want to establish $K(\frac{p}{2}, 4\sigma + \max\{\frac{4(n+1)}{p} - 2n, 0\} + \varepsilon)$ for every $\varepsilon > 0$. By rescaling by a factor of R^2 in both x and t, we may consider families of cylinders of dimensions $R \times \cdots \times R \times R^2$, that point in R^{-1} separated directions. Without loss of generality assume that the central axis of each cylinder in the family makes an angle $\lesssim \pi/4$ with the t axis. By replacing the cylinders by slightly larger ones in a direction that differs by $O(R^{-1})$, and splitting a fat cylinder into $O(1)$ thinner cylinders and using the Minkowski inequality, we may also assume, without loss of generality, that $\mathbb{T} = \left\{\tilde{T}_\theta \colon \theta \in \mathcal{P}_R\right\}$ where for $\theta \in \mathcal{P}_R$,

$$\tilde{T}_\theta := z_\theta + \frac{R^2}{4n} T_\theta;$$

here T_θ is as in (4) and z_θ is an arbitrary point in $\mathbb{R}^{n+1}$. We will also assume that the cylinders intersect, and hence we may assume that $|z_\theta| \leq R^2$ for all $\theta \in \mathcal{P}_R$. For each $\theta \in \mathcal{P}_R$ we will construct a wave packet F_θ so that $\widehat{F_\theta}$ is supported in θ and so that $|F_\theta| \gtrsim 1$ on $\tilde{T}_\theta$. Each F_θ will in turn be the superposition of wave packets whose frequencies are even more localized, via

$$F_\theta(x,t) = \sum_{\beta \in \mathcal{P}_{R^2}, \, \beta \subset \theta} F_\beta(x,t) \tag{8}$$

where each $\widehat{F_\beta}$ is supported in β. We may then apply (1) (with R replaced by R^2) to

$$\sum_{\theta \in \mathcal{P}_R} \varepsilon_\theta a_\theta^{1/2} F_\theta = \sum_{\theta \in \mathcal{P}_R} \sum_{\beta \in \mathcal{P}_{R^2}, \, \beta \subset \theta} \varepsilon_\theta a_\theta^{1/2} F_\beta(x,t) \tag{9}$$

where $\{\varepsilon_\theta\}_{\theta \in \mathcal{P}_R}$ is a random choice of signs ± 1, and $\{a_\theta\}_{\theta \in \mathcal{P}_R}$ is a family of non-negative coefficients; further applying Klintchine's inequality on the left hand side, we will be able to show that

$$\Big\| \sum_{\theta \in \mathcal{P}_R} a_\theta \mathbf{1}_{\tilde{T}_\theta} \Big\|_{L^{p/2}(\mathbb{R}^{n+1})} \lesssim_{p,\sigma,\varepsilon} R^{4\sigma + \max\{\frac{4(n+1)}{p} - 2n, 0\} + \varepsilon} \Big(\sum_{\theta \in \mathcal{P}_R} a_\theta^{p/2} |\tilde{T}_\theta| \Big)^{2/p} \tag{10}$$

for every $\varepsilon > 0$, from which $K(\frac{p}{2}, 4\sigma + \max\{\frac{4(n+1)}{p} - 2n, 0\} + \varepsilon)$ will follow.

To carry this out in detail, let $\Phi(x)$ be as in Lemma 3, and $\eta(t)$ be a Schwartz function on $\mathbb{R}$ whose Fourier transform is compactly supported on $[-1,1]$. In addition, suppose $\Phi(x)$ is real-valued, $\Phi(x) \geq 1$ for $|x| \leq 1$, and $|\eta(t)| \geq 1$ for $|t| \leq [-1,1]$. For $R \gg 1$ and $\beta \in \mathcal{P}_{R^2}$, define

$$\Phi_\beta(x) := \Phi(R^{-2}x) e^{2\pi i x \cdot c_\beta}$$

as in Lemma 3; for $\theta \in \mathcal{P}_R$, let ε_θ be a random sign ± 1 and write z_θ as (y_θ, s_θ). Now define, for each $\theta \in \mathcal{P}_R$, a function $F_\theta(x,t)$ by (8), where for $\beta \in \mathcal{P}_{R^2}$ and $\beta \subset \theta$, $F_\beta(x,t)$ is defined via

$$F_\beta(x - y_\theta, t - s_\theta) := \varepsilon_\theta R^{-n} \eta(R^{-4}t) e^{-\frac{it\Delta}{2\pi}} \Phi_\beta(x).$$

Then the Fourier transform of F_β is

$$\varepsilon_\theta R^{n+4} \widehat{\eta}(R^4(\tau - |\xi|^2)) \widehat{\Phi}(R^2(\xi - c_\beta)) e^{2\pi i (x \cdot y_\theta + t s_\theta)}$$

which is supported on $\mathfrak{R}_\beta$. Furthermore, for each $\theta \in \mathcal{P}_R$,

$$|F_\theta(x,t)| = \left| \sum_{\beta \in \mathcal{P}_{R^2}, \, \beta \subset \theta} F_\beta(x,t) \right| \gtrsim \mathbf{1}_{\tilde{T}_\theta}(x,t). \tag{11}$$

Indeed, by (6),

$$\sum_{\beta\in\mathcal{P}_{R^2},\,\beta\subset\theta} F_\beta(x-y_\theta,t-s_\theta)=\sum_{\beta\in\mathcal{P}_{R^2},\,\beta\subset\theta} R^{-n}\eta(R^{-4}t)e^{2\pi i(x\cdot c_\beta+t|c_\beta|^2)}\Phi(R^{-2}(x+2tc_\beta))+O(R^{-4}t).$$

But we may rewrite the phase in the sum by "Taylor expanding c_β around c_θ", and obtain

$$x\cdot c_\beta+t|c_\beta|^2=x\cdot c_\theta+t|c_\theta|^2+(x+2tc_\theta)\cdot(c_\beta-c_\theta)+t|c_\beta-c_\theta|^2.$$

It follows that

$$\begin{aligned}&\sum_{\beta\in\mathcal{P}_{R^2},\,\beta\subset\theta} F_\beta(x-y_\theta,t-s_\theta)\\ &=e^{2\pi i(x\cdot c_\theta+t|c_\theta|^2)}R^{-n}\eta(R^{-4}t)\sum_{\beta\in\mathcal{P}_{R^2},\,\beta\subset\theta} e^{2\pi i((x+2tc_\theta)\cdot(c_\beta-c_\theta)+t|c_\theta-c_\beta|^2)}\Phi(R^{-2}(x+2tc_\beta))+O(R^{-4}t).\end{aligned}$$

Since $|c_\beta-c_\theta|\le R^{-1}/2$, for any $(x,t)\in\frac{R^2}{4n}T_\theta$, we have

$$\begin{aligned}&\mathrm{Re}\Big(\sum_{\beta\in\mathcal{P}_{R^2},\,\beta\subset\theta} e^{2\pi i((x+2tc_\theta)\cdot(c_\beta-c_\theta)+t|c_\theta-c_\beta|^2)}\Phi(R^{-2}(x+2tc_\beta))\Big)\\ =&\sum_{\beta\in\mathcal{P}_{R^2},\,\beta\subset\theta}\cos(2\pi[(x+2tc_\theta)\cdot(c_\beta-c_\theta)+t|c_\theta-c_\beta|^2])\Phi(R^{-2}(x+2tc_\beta))\\ \ge&\cos\left(\frac{3\pi}{8}\right)\sum_{\beta\in\mathcal{P}_{R^2},\,\beta\subset\theta}\Phi(R^{-2}(x+2tc_\beta))\\ \gtrsim& R^n.\end{aligned}$$

As a result, we obtain

$$\Big|\sum_{\beta\in\mathcal{P}_{R^2},\,\beta\subset\theta} F_\beta(x-y_\theta,t-s_\theta)\Big|\gtrsim\mathbf{1}_{\frac{R^2}{4n}T_\theta}(x,t),$$

verifying (11). We now apply (1) (with R replaced by R^2) to (9), and obtain

$$\Big\|\sum_{\theta\in\mathcal{P}_R}\varepsilon_\theta a_\theta^{1/2}F_\theta(x,t)\Big\|_{L^p(\mathbb{R}^{n+1})}\lesssim_{p,\sigma}(R^2)^\sigma\Big\|\Big(\sum_{\theta\in\mathcal{P}_R}a_\theta\sum_{\beta\in\mathcal{P}_{R^2},\,\beta\subset\theta}|F_\beta(x,t)|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^{n+1})}.$$

Applying Klintchine's inequality to the p-th power of the left hand side, and then taking $p/2$-th root, we obtain

$$\Big\|\sum_{\theta\in\mathcal{P}_R}a_\theta|F_\theta(x,t)|^2\Big\|_{L^{p/2}(\mathbb{R}^{n+1})}\lesssim_{p,\sigma}R^{4\sigma}\Big\|\sum_{\theta\in\mathcal{P}_R}a_\theta\sum_{\beta\in\mathcal{P}_{R^2},\,\beta\subset\theta}|F_\beta(x,t)|^2\Big\|_{L^{p/2}(\mathbb{R}^{n+1})}. \tag{12}$$

By (11), the left hand side of (12) is bounded below by

$$\gtrsim\Big\|\sum_{\theta\in\mathcal{P}_R}a_\theta\mathbf{1}_{\tilde{T}_\theta}\Big\|_{L^{p/2}(\mathbb{R}^{n+1})}$$

By (7), the $L^{p/2}$ norm right hand side of (12) is bounded above by

$$\lesssim_N\Big\|\sum_{\theta\in\mathcal{P}_R}a_\theta\sum_{\beta\in\mathcal{P}_{R^2},\,\beta\subset\theta}R^{-2n}(1+R^{-2}|x+2tc_\beta|+R^{-4}|t|)^{-N}\Big\|_{L^{p/2}(\mathbb{R}^{n+1})} \tag{13}$$

for every positive integer N, because $|z_\theta| \leq R^2$; once we get rid of z_θ, if $N > n+1$, we may rescale and apply Lemma 2 (with R replaced by R^2) and bound (13) by

$$\lesssim_p R^{2\kappa(p/2)} \Big(\sum_{\theta \in \mathcal{P}_R} \sum_{\beta \in \mathcal{P}_{R^2},\, \beta \subset \theta} (a_\theta R^{-2n})^{p/2} R^{2(n+2)} \Big)^{2/p} \simeq R^{2\kappa(p/2)-2n+\frac{4(n+1)}{p}} \Big(\sum_{\theta \in \mathcal{P}_R} a_\theta^{p/2} |\tilde{T}_\theta| \Big)^{2/p},$$

using $R^n R^{2(n+2)} \simeq R^{2(n+1)} |\tilde{T}_\theta|$. As a result, (12) implies

$$\Big\| \sum_{\theta \in \mathcal{P}_R} a_\theta \mathbf{1}_{\tilde{T}_\theta} \Big\|_{L^{p/2}(\mathbb{R}^{n+1})} \lesssim (\log R)^{e(p/2)} R^{4\sigma+2\kappa(p/2)-2n+\frac{4(n+1)}{p}} \Big(\sum_{\theta \in \mathcal{P}_R} a_\theta^{p/2} |\tilde{T}_\theta| \Big)^{2/p}$$

Since $2\kappa(p/2) - 2n + \frac{4(n+1)}{p} = \max\{\frac{4(n+1)}{p} - 2n, 0\}$, this implies $K(\frac{p}{2}, 4\sigma + \max\{\frac{4(n+1)}{p} - 2n, 0\} + \varepsilon)$ for any $\varepsilon > 0$, and that the same holds with $\varepsilon = 0$ if $\frac{p}{2} \neq \frac{n+1}{n}$. $\square$

We now proceed to the proof of Theorem 2. It is well-known that $K(\frac{p}{2}, \kappa)$ implies a corresponding Nikodym maximal estimate; see Tao [4]. We need a small extension of that. We begin with the following lemma.

Lemma 4. *Suppose $1 \leq q \leq \infty$, $\kappa \geq 0$ and $K(q,\kappa)$ holds. For $\theta \in \mathcal{P}_R$, let $w_\theta \in \mathbb{R}^n$ be an arbitrary vector with $|w_\theta| \leq 2$, and let*

$$T'_\theta := (c_\theta, 0) + \{(x,t) \in \mathbb{R}^{n+1} \colon |x + t w_\theta| \leq R^{-1}, |t| \leq 1\}$$

where c_θ is the center of the square θ. Then for any non-negative coefficients $\{a_\theta\}_{\theta \in \mathcal{P}_R}$, we have

$$\Big\| \sum_{\theta \in \mathcal{P}_R} a_\theta \mathbf{1}_{T'_\theta} \Big\|_{L^q(\mathbb{R}^{n+1})} \lesssim_{q,\kappa} R^\kappa \Big(\sum_{\theta \in \mathcal{P}_R} a_\theta^q |T'_\theta| \Big)^{1/q}. \tag{14}$$

More generally, let T^*_θ be the infinite cylinder

$$T^*_\theta := (c_\theta, 0) + \{(x,t) \in \mathbb{R}^{n+1} \colon |x + t w_\theta| \leq R^{-1}\}.$$

Then for $N \geq \frac{n+2}{q}$,

$$\Big\| \sum_{\theta \in \mathcal{P}_R} a_\theta (1+|t|)^{-N} \mathbf{1}_{T^*_\theta} \Big\|_{L^q(\mathbb{R}^{n+1})} \lesssim_{q,\kappa} R^\kappa \Big(\sum_{\theta \in \mathcal{P}_R} a_\theta^q |T'_\theta| \Big)^{1/q}. \tag{15}$$

The following proof is essentially in Tao [4].

Proof. Suppose $1 \leq q \leq \infty$, $\kappa \geq 0$ and $K(q,\kappa)$ holds. For $\theta \in \mathcal{P}_R$, let $T'_{\theta,\text{top}}$ be the intersection of T'_θ with the strip $\{1/2 \leq |t| \leq 1\}$ in $\mathbb{R}^{n+1}$. We first show that

$$\int_{\mathbb{R}^{n+1}} \Big(\sum_{\theta \in \mathcal{P}_R} a_\theta \mathbf{1}_{T'_{\theta,\text{top}}}(x,t) \Big)^q dxdt \lesssim_{q,\kappa} R^{\kappa q} \sum_{\theta \in \mathcal{P}_R} a_\theta^q |T'_\theta|. \tag{16}$$

Indeed, we perform a projective change of variables $(y,s) := I(x,t)$ where

$$I(x,t) := (x/t, 1/t).$$

The Jacobian is $\lesssim 1$ on the support of the integrand, and

$$I(T'_{\theta,\text{top}}) = (-w_\theta, 0) + \{(y,s) \in \mathbb{R}^{n+1} \colon |y - s c_\theta| \leq |s| R^{-1}, 1 \leq |s| \leq 2\} \subset \bar{T}_\theta$$

where

$$\bar{T}_\theta := (-w_\theta, 0) + 2\{(y,s) \in \mathbb{R}^{n+1} \colon |y - sc_\theta| \le R^{-1}, |s| \le 1\}$$

is a cylinder of dimensions $\simeq R^{-1} \times \cdots \times R^{-1} \times 1$, pointing in the direction $(c_\theta, 1)$; such directions are $\simeq R^{-1}$ separated as θ varies over $\mathcal{P}_R$. Thus

$$\int_{\mathbb{R}^{n+1}} \Big(\sum_{\theta \in \mathcal{P}_R} a_\theta \mathbf{1}_{T'_{\theta,\text{top}}}(x,t) \Big)^q dxdt \lesssim \int_{\mathbb{R}^{n+1}} \Big(\sum_{\theta \in \mathcal{P}_R} a_\theta \mathbf{1}_{\bar{T}_\theta}(y,s) \Big)^q dyds$$

and (16) follows from our assumption $K(q,\kappa)$.

Next, for $k \ge 1$, let $T'_{\theta,k}$ be the intersection of T'_θ with the strip $\{2^{-(k+1)} \le |t| \le 2^{-k}\}$ in $\mathbb{R}^{n+1}$. We show that

$$\int_{\mathbb{R}^{n+1}} \Big(\sum_{\theta \in \mathcal{P}_R} a_\theta \mathbf{1}_{T'_{\theta,k}}(x,t) \Big)^q dxdt \lesssim_{q,\kappa} 2^{-k} R^{\kappa q} \sum_{\theta \in \mathcal{P}_R} a_\theta^q |T'_\theta|. \tag{17}$$

We perform a change of variables by dilation in the t variable, via $t' = 2^k t$ so that the left hand side of (17) becomes

$$2^{-k} \int_{\mathbb{R}^{n+1}} \Big(\sum_{\theta \in \mathcal{P}_R} a_\theta \mathbf{1}_{T''_{\theta,k}}(x,t') \Big)^q dxdt' \tag{18}$$

where $T''_{\theta,k}$ is the cylinder $T'_{\theta,k}$ dilated by 2^k in the t direction, i.e.

$$T''_{\theta,k} = (c_\theta, 0) + \{(x,t') \in \mathbb{R}^{n+1} \colon |x + t'(2^{-k} w_\theta)| \le R^{-1}, 1/2 \le |t'| \le 1\}.$$

Note that $|2^{-k} w_\theta| \le 1$ since $|w_\theta| \le 1$. Thus (16) implies that (18) is bounded by the right hand side of (17), as desired.

Summing (16) and (17) over $k \in \mathbb{N}$, and then taking q-th root, we obtain (14).

An easy modification of the above proof gives also (15). Indeed, for $\theta \in \mathcal{P}_R$, let $T^*_{\theta,\text{top}}$ be the intersection of T^*_θ with the strip $\{|t| \ge 1/2\}$ in $\mathbb{R}^{n+1}$. Then for $N \ge (n+2)/q$, we show that

$$\int_{\mathbb{R}^{n+1}} \Big(\sum_{\theta \in \mathcal{P}_R} a_\theta |t|^{-N} \mathbf{1}_{T^*_{\theta,\text{top}}}(x,t) \Big)^q dxdt \lesssim_{q,\kappa} R^{\kappa q} \sum_{\theta \in \mathcal{P}_R} a_\theta^q |T'_\theta|. \tag{19}$$

To see this, we perform the same projective change of variables $(y,s) := I(x,t)$. This time it will be crucial that $dxdt = s^{-(n+2)} dyds$, and we still have $I(T^*_{\theta,\text{top}}) \subset T_{\theta,\text{top}}$ because

$$I(T^*_{\theta,\text{top}}) = (-w_\theta, 0) + \{(y,s) \in \mathbb{R}^{n+1} \colon |y - sc_\theta| \le |s| R^{-1}, |s| \le 2\}.$$

Thus

$$\int_{\mathbb{R}^{n+1}} \Big(\sum_{\theta \in \mathcal{P}_R} a_\theta |t|^{-N} \mathbf{1}_{T^*_{\theta,\text{top}}}(x,t) \Big)^q dxdt \lesssim \int_{\mathbb{R}^{n+1}} \Big(\sum_{\theta \in \mathcal{P}_R} a_\theta \mathbf{1}_{T_{\theta,\text{top}}}(y,s) \Big)^q s^{Nq-(n+2)} dyds,$$

and the factor $s^{Nq-(n+2)}$ can be bounded by $2^{Nq-(n+2)} \lesssim 1$ when $N \ge (n+2)/q$. (19) then follows. Together with (17), we obtain the desired conclusion (15). $\square$

We need a slightly more general version of Lemma 4, where we allow infinitely many cylinders rather than just $\simeq R^n$ cylinders based on $[-1,1]^n \times \{0\}$.

Lemma 5. *Suppose $1 \le q \le \infty$, $\kappa \ge 0$, and $K(q,\kappa)$ holds. For $\mu \in R^{-1}\mathbb{Z}^n$, let $w_\mu \in \mathbb{R}^n$ be an arbitrary vector with $|w_\mu| \le 2$. Then any non-negative coefficients $\{a_\mu\}_{\mu\in R^{-1}\mathbb{Z}^n}$ and any $N > 2n + \frac{n+2}{q}$, we have*

$$\Big\| \sum_{\mu\in R^{-1}\mathbb{Z}^n} a_\mu (1 + R|x-\mu+tw_\mu| + |t|)^{-N} \Big\|_{L^q(\mathbb{R}^{n+1})} \lesssim_{q,\kappa} R^\kappa \Big(\sum_{\mu\in R^{-1}\mathbb{Z}^n} a_\mu^q R^{-n} \Big)^{1/q}. \tag{20}$$

Proof. For $\mu \in R^{-1}\mathbb{Z}^n$ let T_μ^* be the infinite cylinder

$$T_\mu^* := (\mu, 0) + \{(x,t) \in \mathbb{R}^{n+1} \colon |x + tw_\mu| \le R^{-1}\}.$$

We first show that for $N > n + \frac{n+2}{q}$, we have

$$\Big\| \sum_{\mu\in R^{-1}\mathbb{Z}^n} a_\mu (1+|t|)^{-N} \mathbf{1}_{T_\mu^*} \Big\|_{L^q(\mathbb{R}^{n+1})} \lesssim_{q,\kappa} R^\kappa \Big(\sum_{\mu\in R^{-1}\mathbb{Z}^n} a_\mu^q R^{-n} \Big)^{1/q}. \tag{21}$$

For $m \in \mathbb{Z}^{n+1}$, let Q_m be the unit cube in $\mathbb{R}^{n+1}$ centered at m. If $m = (m', m'')$ then for T_μ^* to intersect Q_m we must have $|\mu - m'| \lesssim 1 + |m''|$. Thus

$$\Big\| \sum_{\mu\in R^{-1}\mathbb{Z}^n} a_\mu (1+|t|)^{-\frac{n+2}{q}} \mathbf{1}_{T_\mu^*} \Big\|_{L^q(Q_m)} \lesssim \sum_{n'\in\mathbb{Z}^n,\, |n'-m'|\lesssim 1+|m''|} \Big\| \sum_{\mu\in R^{-1}\mathbb{Z}^n,\, |\mu-n'|\le 1} a_\mu (1+|t|)^{-\frac{n+2}{q}} \mathbf{1}_{T_\mu^*} \Big\|_{L^q(Q_m)}$$

which by (15) is

$$\begin{aligned} &\lesssim_{q,\kappa} R^\kappa \sum_{n'\in\mathbb{Z}^n,\, |n'-m'|\lesssim 1+|m''|} \Big(\sum_{\mu\in R^{-1}\mathbb{Z}^n,\, |\mu-n'|\le 1} a_\mu^q R^{-n} \Big)^{1/q} \\ &\lesssim R^\kappa \Big(\sum_{\mu\in R^{-1}\mathbb{Z}^n,\, |\mu-m'|\lesssim 1+|m''|} a_\mu^q R^{-n} \Big)^{1/q} (1+|m''|)^{n(q-1)/q}. \end{aligned}$$

As a result, raising both sides to power q and summing over $m' \in \mathbb{Z}^n$, we obtain

$$\Big\| \sum_{\mu\in R^{-1}\mathbb{Z}^n} a_\mu (1+|t|)^{-\frac{n+2}{q}} \mathbf{1}_{T_\mu^*} \Big\|_{L^q(\mathbb{R}^n \times (m'' + [-1/2, 1/2]))} \lesssim_{q,\kappa} (1+|m''|)^n R^\kappa \Big(\sum_{\mu\in R^{-1}\mathbb{Z}^n} a_\mu^q R^{-n} \Big)^{1/q}$$

from which (21) follows upon multiplying by $(1+|m''|)^{-(N-\frac{n+2}{q})}$ and then summing over m''. Finally, the left hand side of (20) is bounded by

$$\sum_{m'\in\mathbb{Z}^n} (1+|m'|)^{-(N-N')} \Big\| \sum_{\mu\in R^{-1}\mathbb{Z}^n} a_\mu (1+|t|)^{-N'} \mathbf{1}_{m'(R^{-1},0)+T_\mu^*} \Big\|_{L^q(\mathbb{R}^{n+1})}.$$

The L^q norm above is independent of m', and is controlled by (21) as long as $N' > n + \frac{n+2}{q}$. The sum over m' is finite as long as $N - N' > n$. Thus (20) follows when $N > 2n + \frac{n+2}{q}$. ∎

Lemma 5 can be reformulated in terms of a Nikodym maximal function via duality. For $R \gg 1$ and $w \in \mathbb{R}^n$ with $|w| \le 2$, let

$$T_w := \{(x,t) \in \mathbb{R}^{n+1} \colon |t| \le 1, |x + tw| \le R^{-1}\}$$

note that $|T_w| \simeq R^{-n}$ uniformly in w. Let $\mathfrak{N}_R$ be the Nikodym maximal function, defined by

$$\mathfrak{N}_R g(y) := \sup_{w\in\mathbb{R}^n,\, |w|\le 2} \frac{1}{R^{-n}} \int_{(y,0)+T_w} |g(x,t)| dxdt, \quad y \in \mathbb{R}^n.$$

More generally, for $N > 2n + \frac{n+2}{q}$, let

$$\mathfrak{N}_R^* g(y) := \sup_{w\in\mathbb{R}^n,\, |w|\le 2} \frac{1}{R^{-n}} \int_{\mathbb{R}^{n+1}} |g(x,t)|(1+R|x-y+tw|+|t|)^{-N}dxdt, \quad y\in\mathbb{R}^n.$$

Lemma 6. *Suppose $1\le q\le\infty$, $\kappa\ge 0$ and $K(q,\kappa)$ holds. Let $q'=q/(q-1)$ be the conjugate exponent to q. Then for any $R\gg 1$,*

$$\|\mathfrak{N}_R g\|_{L^{q'}(\mathbb{R}^n)} \lesssim_{q,\kappa} R^\kappa \|g\|_{L^{q'}(\mathbb{R}^{n+1})} \tag{22}$$

and for $N > 2n + \frac{n+2}{q}$,

$$\|\mathfrak{N}_R^* g\|_{L^{q'}(\mathbb{R}^n)} \lesssim_{q,\kappa} R^\kappa \|g\|_{L^{q'}(\mathbb{R}^{n+1})}. \tag{23}$$

Proof. Since $\mathfrak{N}_R g(y) \lesssim \mathfrak{N}_R^* g(y)$, clearly (23) implies (22). Thus we prove only (23).

For $\mu\in R^{-1}\mathbb{Z}^n$ we have

$$\mathfrak{N}_R^* g(y) \simeq \mathfrak{N}_R^* g(\mu)$$

for every $y\in\mathbb{R}^n$ with $|y-\mu|\le R^{-1}$. This is because for such y's, we have

$$(1+R|x-y+tw|+|t|)^{-N} \simeq (1+R|x-\mu+tw|+|t|)^{-N}.$$

Thus

$$\|\mathfrak{N}_R^* g\|_{L^{q'}(\mathbb{R}^n)} \simeq \Big(\sum_{\mu\in R^{-1}\mathbb{Z}^n} \mathfrak{N}_R^* g(\mu)^{q'} R^{-n}\Big)^{1/q}.$$

To compute the latter, let $\{a_\mu\}_{\mu\in R^{-1}\mathbb{Z}^n}$ so that $\sum_{\mu\in R^{-1}\mathbb{Z}^n} a_\mu^q R^{-n}=1$. Then picking $w_\mu\in\mathbb{R}^n$ with $|w_\mu|\le 2$ so that

$$\mathfrak{N}_R^* g(\mu) \simeq \frac{1}{R^{-n}}\int_{\mathbb{R}^{n+1}} |g(x,t)|(1+R|x-\mu+tw_\mu|+|t|)^{-N}dxdt,$$

we have

$$\sum_{\mu\in R^{-1}\mathbb{Z}^n} a_\mu \mathfrak{N}_R^* g(\mu) R^{-n} = \int_{\mathbb{R}^{n+1}} |g(x,t)| \sum_{\mu\in R^{-1}\mathbb{Z}^n} a_\mu (1+R|x-\mu+tw_\mu|+|t|)^{-N}dxdt$$

which by Hölder and Lemma 5 is bounded by

$$\lesssim_{q,\kappa} R^\kappa \|g\|_{L^q(\mathbb{R}^{n+1})}$$

if $N > 2n+\frac{n+2}{q}$. This completes the proof of (23). □

Proof of Theorem 2. Let $2\le p\le\infty$, $\sigma\ge 0$ and $\kappa\ge 0$ be such that $RS(p,\sigma)$ and $K(\frac{p}{2},\kappa)$ holds. The local smoothing estimate $LS(p,\sigma+\frac{\kappa}{2})$ can now be deduced in a few strokes.

First, by rescaling, let f be a Schwartz function on $\mathbb{R}^n$ whose Fourier transform is supported on the annulus $\{1/2\le|\xi|\le 1\}$. Then $LS(p,\sigma+\frac{\kappa}{2})$ will follow if we can show that

$$\|e^{-\frac{it\Delta}{2\pi}} f\|_{L^p(\mathbb{R}^n\times[0,R^2])} \lessapprox R^{\frac{2}{p}} R^{\sigma+\frac{\kappa}{2}} \|f\|_{L^p(\mathbb{R}^n)}. \tag{24}$$

To prove (24) we decompose f as follows. Let φ be a smooth function with compact support on $[-1,1]^n$ so that

$$\sum_{\nu\in\mathbb{Z}^n}\varphi(\xi-\nu)=1 \quad \text{for all } \xi\in\mathbb{R}^n.$$

Then for any $|\xi|\simeq 1$, we have

$$\sum_{\theta\in\mathcal{P}_R}\varphi(R(\xi-c_\theta))=1 \tag{25}$$

where c_θ is the center of the square θ. If $\theta\in\mathcal{P}_R$, we define f_θ to be the Schwartz function given by

$$\widehat{f_\theta}(\xi):=\varphi(R(\xi-c_\theta))\widehat{f}(\xi);$$

(25) then gives

$$f=\sum_{\theta\in\mathcal{P}_R}f_\theta.$$

Let $\eta(t)$ be a Schwartz function on $\mathbb{R}$ so that $|\eta(t)|\ge 1$ for all $t\in[0,1]$, and so that $\widehat{\eta}(\tau)$ is supported on $[-1,1]$. Then for $t\in[0,R^2]$,

$$|e^{-\frac{it\Delta}{2\pi}}f(x)|\le\Big|\sum_{\theta\in\mathcal{P}_R}\eta(R^{-2}t)e^{-\frac{it\Delta}{2\pi}}f_\theta(x)\Big|,$$

and for each $\theta\in\mathcal{P}_R$, the function $\eta(R^{-2}t)e^{-\frac{it\Delta}{2\pi}}f_\theta(x)$ has (space-time) Fourier transform

$$R^2\widehat{\eta}(R^2(\tau-|\xi|^2))\varphi(R(\xi-c_\theta))\widehat{f}(\xi)$$

which is supported in $\mathfrak{R}_\theta$. Thus from our assumption $RS(p,\sigma)$, we obtain that

$$\|e^{-\frac{it\Delta}{2\pi}}f(x)\|_{L^p(\mathbb{R}^n\times[0,R^2])}\lesssim_{p,\sigma}R^\sigma\Big\|\Big(\sum_{\theta\in\mathcal{P}_R}|\eta(R^{-2}t)e^{-\frac{it\Delta}{2\pi}}f_\theta(x)|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^{n+1})}. \tag{26}$$

Next, we use our assumption $K(\frac{p}{2},\kappa)$ to deduce that

$$\Big\|\Big(\sum_{\theta\in\mathcal{P}_R}|\eta(R^{-2}t)e^{-\frac{it\Delta}{2\pi}}f_\theta(x)|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^{n+1})}\lesssim_{p,\kappa}R^{\frac{2}{p}}R^{\frac{\kappa}{2}}\Big\|\Big(\sum_{\theta\in\mathcal{P}_R}|f_\theta(x)|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}. \tag{27}$$

To see this, let Φ be as in Lemma 3, and satisfy additionally the assumption that $\widehat{\Phi}=1$ on the support of φ. Then $e^{-\frac{it\Delta}{2\pi}}f_\theta(x)=R^{-n}f_\theta * e^{-\frac{it\Delta}{2\pi}}\Phi_\theta(x)$, so from the upper bound (7) of $|\eta(R^{-2}t)e^{-\frac{it\Delta}{2\pi}}\Phi_\theta(x)|$, we obtain that

$$|\eta(R^{-2}t)e^{-\frac{it\Delta}{2\pi}}f_\theta(x)|\lesssim_N\int_{\mathbb{R}^n}|f_\theta(y)|R^{-n}(1+R^{-1}|x-y+2tc_\theta|+R^{-2}|t|)^{-N}dy$$

for every positive integer N. Cauchy-Schwarz then gives

$$|\eta(R^{-2}t)e^{-\frac{it\Delta}{2\pi}}f_\theta(x)|^2\lesssim_N\int_{\mathbb{R}^n}|f_\theta(y)|^2R^{-n}(1+R^{-1}|x-y+2tc_\theta|+R^{-2}|t|)^{-N}dy.$$

To estimate the right hand side of (26), let q' be the dual exponent of $q:=p/2$, and let $g\in L^{q'}(\mathbb{R}^{n+1})$ with $\|g\|_{L^{q'}(\mathbb{R}^{n+1})}=1$. We estimate

$$\int_{\mathbb{R}^{n+1}}\sum_{\theta\in\mathcal{P}_R}|\eta(R^{-2}t)e^{-\frac{it\Delta}{2\pi}}f_\theta(x)|^2g(x,t)dxdt\lesssim R^2\int_{\mathbb{R}^n}\sum_{\theta\in\mathcal{P}_R}|f_\theta(y)|^2\mathfrak{N}_R^{**}g(y)dy \tag{28}$$

where

$$\mathfrak{N}_R^{**}g(y) := \sup_{\theta\in\mathcal{P}_R}\int_{\mathbb{R}^{n+1}}|g(x,t)|R^{-(n+2)}(1+R^{-1}|x-y+2tc_\theta|+R^{-2}|t|)^{-N}dxdt$$

is a rescaled version of $\mathfrak{N}_R^*$. Indeed, applying Lemma 6 to $g(R^2x,R^2t)$ instead of $g(x,t)$, we obtain,

$$\|\mathfrak{N}_R^{**}\|_{L^{q'}(\mathbb{R}^{n+1})\to L^{q'}(\mathbb{R}^n)} = R^{-\frac{2}{q'}}\|\mathfrak{N}_R^*\|_{L^{q'}(\mathbb{R}^{n+1})\to L^{q'}(\mathbb{R}^n)} \lesssim_{q,\kappa} R^{-\frac{2}{q'}}R^\kappa.$$

if $N > 2n+\frac{n+2}{q}$. Thus from (28), we obtain

$$\Big(\int_{\mathbb{R}^{n+1}}\Big(\sum_{\theta\in\mathcal{P}_R}|\eta(R^{-2}t)e^{-\frac{it\Delta}{2\pi}}f_\theta(x)|^2\Big)^{p/2}dxdt\Big)^{2/p} \lesssim_{p,\kappa} R^2R^{-\frac{2}{q'}}R^\kappa\Big(\int_{\mathbb{R}^n}\Big(\sum_{\theta\in\mathcal{P}_R}|f_\theta(x)|^2\Big)^{p/2}dx\Big)^{2/p}.$$

(27) follows by taking square roots of both sides, and recalling that q' is the dual exponent of $p/2$.

Finally, it remains to observe that for any $2\le p<\infty$, we have

$$\Big\|\Big(\sum_{\theta\in\mathcal{P}_R}|f_\theta|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}, \tag{29}$$

which follows from the following lemma by rescaling by R in the frequency space; combining (26), (27) and (29) we have our desired local smoothing estimate (24) and hence $LS(p,\sigma+\frac{\kappa}{2})$. □

Lemma 7. *For any $2\le p\le\infty$, we have*

$$\Big\|\Big(\sum_{\nu\in\mathbb{Z}^n}|f*\Phi_\nu|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$$

where $\widehat{\Phi_\nu}(\xi) := \varphi(\xi-\nu)$ and φ is a Schwartz function on $\mathbb{R}^n$.

Proof of Lemma 7. The following proof appears, for instance, in Córdoba [2].

The idea is to write

$$\sum_{\nu\in\mathbb{Z}^n}|f*\Phi_\nu(x)|^2 = \int_{[0,1]^n}\Big|\sum_{\nu\in\mathbb{Z}^n}f*\Phi_\nu(x)e^{2\pi i\nu\cdot y}\Big|^2dy$$

using Parseval. Since

$$f*\Phi_\nu(x)e^{2\pi i\nu\cdot y} = \int_{\mathbb{R}^n}\widehat{f}(\xi)\varphi(\xi-\nu)e^{2\pi ix\cdot\xi}e^{2\pi i\nu\cdot y}d\xi$$

and Poisson summation gives

$$\sum_{\nu\in\mathbb{Z}^n}\varphi(\xi-\nu)e^{2\pi i\nu\cdot y} = \sum_{\nu\in\mathbb{Z}^n}\Phi_0(y+\nu)e^{-2\pi i(y+\nu)\cdot\xi},$$

we obtain

$$\sum_{\nu\in\mathbb{Z}^n}f*\Phi_\nu(x)e^{2\pi i\nu\cdot y} = \sum_{\nu\in\mathbb{Z}^n}f(x-y-\nu)\Phi_0(y+\nu).$$

Hence by Cauchy-Schwarz,

$$\Big|\sum_{\nu\in\mathbb{Z}^n}f*\Phi_\nu(x)e^{2\pi i\nu\cdot y}\Big|^2 \lesssim \sum_{\nu\in\mathbb{Z}^n}|f(x-y-\nu)|^2|\Phi_0(y+\nu)|,$$

which yields

$$\int_{[0,1]^n} \Big| \sum_{\nu\in\mathbb{Z}^n} f * \Phi_\nu(x) e^{2\pi i \nu\cdot y} \Big|^2 dy \lesssim \int_{\mathbb{R}^n} |f(x-z)|^2 |\Phi_0(z)| dz = |f|^2 * |\Phi_0|(x).$$

It follows that

$$\Big\| \Big(\sum_{\nu\in\mathbb{Z}^n} |f * \Phi_\nu|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^n)} \lesssim \Big\| |f|^2 * |\Phi_0| \Big\|_{L^{p/2}(\mathbb{R}^n)}^{1/2} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$$

when $2 \le p \le \infty$. □

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