Average theorem, Restriction theorem and Strichartz estimates

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Abstract

We provide the details of the proof of the average theorem and the restriction theorem. Emphasis has been placed on the relation between the decay of the Fourier transform of the measure carried on a submanifold and the gain in regularity. The relation between the restriction theorem and a Strichartz estimate is also explained.

Theorem 1 (Average theorem)**.** *Let S be a smooth submanifold of $\mathbb{R}^n$, $d\sigma$ be the induced Lebesgue measure on it, and $d\mu(x) = \eta(x)d\sigma(x)$ where $\eta \in C_c^\infty(\mathbb{R}^n)$. Suppose for some $\alpha > 0$*

$$\widehat{d\mu}(\xi) = O(|\xi|^{-\alpha})$$

as $|\xi| \to \infty$. Define the average operator by

$$Af(x) = \int_S f(x-y)d\mu(y).$$

Then

(a) A maps $L^2(\mathbb{R}^n)$ to $L^2_\alpha(\mathbb{R}^n)$.

(b) A maps $L^{\frac{2\alpha+2}{2\alpha+1}}(\mathbb{R}^n)$ to $L^{2\alpha+2}(\mathbb{R}^n)$.

Proof. The proof of (a) is easy; just observe that if $|\widehat{d\mu}(\xi)| \le C|\xi|^{-\alpha}$, then together with the trivial bound $|\widehat{d\mu}(\xi)| \le C$ that holds by the finiteness of the measure $d\mu$, we have

$$|\widehat{d\mu}(\xi)| \le C(1+|\xi|^2)^{-\alpha/2}.$$

Thus

$$\begin{aligned}\int_{\mathbb{R}^n}(1+|\xi|^2)^\alpha|\widehat{Af}(\xi)|^2d\xi &= \int_{\mathbb{R}^n}(1+|\xi|^2)^\alpha|\widehat{f}(\xi)|^2|\widehat{d\mu}(\xi)|^2d\xi \\ &\le C^2\int_{\mathbb{R}^n}|\widehat{f}(\xi)|^2d\xi,\end{aligned}$$

which implies that

$$\|Af\|_{L^2_\alpha} \leq C\|f\|_{L^2}.$$

To prove (b), we proceed in a number of steps.

Localizing the average operator

First we localize the average operator as follows. By means of a partition of unity, we may assume without loss of generality that the support of $d\mu$ lies in a coordinate chart U of S (the average operator is a finite sum of such in any case). By a change of coordinate, restricting to a smaller coordinate patch if necessary, we may assume that S is the graph of a function $F(x')$ there, i.e. S is defined by

$$S = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \colon x_n = F(x')\}.$$

Then

$$d\mu(x) = \eta_1(x')\sqrt{1 + |\nabla F(x')|^2} dx'$$

there, where η_1 is a smooth function with compact support on $\mathbb{R}^{n-1}$. For simplicity, we write this as

$$d\mu(x) = \phi(x')dx'$$

with $\phi \in C_c^\infty(\mathbb{R}^{n-1})$. In this local coordinate system,

$$Af(x) = \int_{\mathbb{R}^{n-1}} f(x' - y', x_n - F(y'))\phi(y')dy'.$$

The idea is that we can embed it in an analytic family of operators and apply the complex interpolation theorem, which we state as follows:

Theorem 2 (Interpolation of Operators)**.** *Suppose $\alpha > 0$, and that for each complex number s in the strip $-\alpha \leq \Re(s) \leq 1$, T_s is a linear operator defined on simple functions on $\mathbb{R}^n$ such that for any fixed simple functions f and g, $\Phi(s) = \int T_s f(x)g(x)dx$ is analytic and bounded (as a function of s) in the open strip $-\alpha < \Re(s) < 1$ and is continuous on the closure of the strip. Suppose further that there are exponents $p_0, p_1, q_0, q_1 \in [1, \infty]$ such that*

$$\begin{cases} \|T_{-\alpha+it}f\|_{L^{q_0}} \leq \|f\|_{L^{p_0}} \\ \|T_{1+it}f\|_{L^{q_1}} \leq \|f\|_{L^{p_1}} \end{cases}$$

uniformly for $t \in \mathbb{R}$. Then

$$\|T_0 f\|_{L^q} \leq \|f\|_{L^p}$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \text{and} \quad \theta = \frac{\alpha}{\alpha+1}.$$

In particular, T_0 extends uniquely to a bounded operator $L^p \to L^q$ with norm at most 1.

Embedding in an analytic family of operators

To embed A in an analytic family of operators, first observe that Af can be written as

$$Af(x) = \int_{\mathbb{R}^n} f(x-y)\delta_0(y_n - F(y'))\phi(y')dy.$$

It is profitable to introduce a cut-off function ψ in the y_n variable; indeed

$$Af(x) = \int_{\mathbb{R}^n} f(x-y)\delta_0(y_n - F(y'))\psi(y_n - F(y'))\phi(y')dy,$$

where $\psi \in C_c^\infty(\mathbb{R})$ with $\psi \equiv 1$ near 0 and $\psi = 0$ outside $(-1,1)$. The point is that there is a way to embed the distribution δ_0 on $\mathbb{R}$ into an analytic family of distributions. Consider the distribution

$$g \in \mathcal{S}(\mathbb{R}) \mapsto j_s(g) := \int_{\mathbb{R}} \frac{1}{\Gamma(s)} t_+^{s-1} g(t)dt,$$

where $t_+ = \max\{t, 0\}$ and s is a complex parameter. The integral converges if Re $s > 0$. It can be analytically continued to Re $s > -1$ by integration by parts; indeed for Re $s > 0$,

$$j_s(g) = \int_{\mathbb{R}} \frac{1}{\Gamma(s)} t_+^{s-1} g(t)dt = \int_0^\infty \frac{1}{s\Gamma(s)} \frac{d}{dt}(t^s) g(t)dt = -\int_0^\infty \frac{1}{\Gamma(s+1)} t^s g'(t)dt,$$

the last of which converges if Re $s > -1$. Hence we can take the last integral as the definition of j_s when Re $s > -1$. In particular, when $s = 0$, the distribution is

$$j_0(g) = -\int_0^\infty \frac{1}{\Gamma(1)} t^0 g'(t)dt = -\int_0^\infty g'(t)dt = g(0),$$

hence j_0 is just δ_0. More generally, we can define the distribution for Re $s > -k$ by the (convergent) integral

$$j_s(g) = (-1)^k \int_{\mathbb{R}} \frac{1}{\Gamma(s+k)} t^{s+k-1} g^{(k)}(t)dt.$$

This gives us an analytic family of distributions, in the sense that $j_s(g)$ is a holomorphic function of s for each Schwartz function g on $\mathbb{R}$. As a result, we are tempted to define an analytic family of operators by

$$T_s f(x) = \int_{\mathbb{R}^n} f(x-y) j_s(y_n - F(y'))\psi(y_n - F(y'))\phi(y')dy,$$

say for Schwartz functions f, with j_s acting on the y_n variable. This can be written as

$$T_s f = f * K_s,$$

where K_s is the distribution defined by

$$K_s(y) = j_s(y_n - F(y'))\psi(y_n - F(y'))\phi(y'). \tag{1}$$

For Re $s > 0$, $K_s(y)$ is just the function given by

$$K_s(y) = \frac{1}{\Gamma(s)}(y_n - F(y'))_+^{s-1}\psi(y_n - F(y'))\phi(y').$$

(Here we already see why it was good to incorporate ψ in our formula for A; it is only by introducing this harmless factor that K_s has compact support in the y_n variable, or at least has some decay as $y_n \to \infty$.) Note also that the distribution K_0 is just $d\mu$.

Boundedness of K_s

Now when Re $s = 1$, $K_s(y)$ is a nice bounded function of y; indeed then

$$|K_s(y)| \leq \frac{C}{\Gamma(s)},$$

and hence T_s maps L^1 to L^∞ with norm at most $C\Gamma(s)^{-1}$. Note that

$$\frac{1}{\Gamma(s)} = \frac{\sin \pi s}{\pi}\Gamma(1-s),$$

so when Re $s = 1$,

$$\frac{1}{\Gamma(s)} \leq Ce^{\text{Im } s},$$

which grows exponentially with Im s as Im $s \to \infty$. As a result, the maps T_s does not quite satisfy the conditions of the interpolation that we stated above; this can be taken care of easily though, so let's not worry about it for now.

Fourier transform of K_s

Next we would like to obtain an L^2 theory for T_s for appropiate values of s. Thus we need to compute and estimate $\widehat{K_s}(\xi)$. First, suppose Re $s > 0$, so that $K_s(y)$ is a nice integrable function of y. Then

$$\begin{aligned}
&\widehat{K_s}(\xi)\\
=&\frac{1}{\Gamma(s)}\int_{\mathbb{R}^n}(y_n - F(y'))_+^{s-1}\psi(y_n - F(y'))\phi(y')e^{-2\pi i y\cdot\xi}dy\\
=&\frac{1}{\Gamma(s)}\int_{\mathbb{R}^{n-1}}\left(\int_{\mathbb{R}}(y_n - F(y'))_+^{s-1}\psi(y_n - F(y'))e^{-2\pi i y_n\xi_n}dy_n\right)\phi(y')e^{-2\pi i y'\cdot\xi'}dy'\\
=&\frac{1}{\Gamma(s)}I(s,\xi_n)\int_{\mathbb{R}^{n-1}}\phi(y')e^{-2\pi i(y'\cdot\xi' + F(y')\xi_n)}dy'\\
=&\frac{1}{\Gamma(s)}I(s,\xi_n)\widehat{d\mu}(\xi) \qquad (2)
\end{aligned}$$

where

$$I(s,\xi_n) = \int_{\mathbb{R}} t_+^{s-1}\psi(t)e^{-2\pi i t\xi_n}dt;$$

this follows from the change of variable $y_n - F(y') \mapsto y_n$ and that $d\mu(y) = \phi(y')dy'$. Now when Re $s \leq 0$, K_s is a distribution with compact support, so $\widehat{K_s}(\xi)$ is the function defined by $\langle e^{-2\pi i y\cdot\xi}, K_s(y)\rangle$ which depends analytically on s. Furthermore, $\Gamma(s)^{-1}I(s,\xi_n)$ also has an analytic continuation to the right half plane by integration by parts, just as we analytically continued the distribution j_s. Since both sides of (2) can be analytically continued to the whole complex plane, the identity continues to hold even when Re $s \le 0$, and we shall see that $\widehat{K_s}(\xi)$ is a bounded function of ξ on the vertical line Re $s = -\alpha$, thereby giving us the desired L^2 theory of T_s that allows us to apply complex interpolation.

Estimating $I(s,\xi_n)$

Now we estimate $I(s,\xi_n)$: first for Re $s > 0$ we have the trivial bound

$$|I(s,\xi_n)| \le \int_{\mathbb{R}} t_+^{\mathrm{Re}\, s-1}|\psi(t)|dt \le C\int_0^1 t^{\mathrm{Re}\, s-1}dt \le C_{\mathrm{Re}\, s}. \tag{3}$$

Actually via an integration by parts, we can obtain a better decay for large $|\xi_n|$ if Re $s > 1$: in that case

$$\begin{aligned} I(s,\xi_n) &= \int_0^\infty t^{s-1}\psi(t)\frac{1}{-2\pi i\xi_n}\frac{d}{dt}\left(e^{-2\pi i t\xi_n}\right)dt \\ &= \frac{1}{2\pi i\xi_n}\int_0^\infty \frac{d}{dt}(t^{s-1}\psi(t))e^{-2\pi i t\xi_n}dt \end{aligned}$$

the boundary terms vanishing since Re $s > 1$ and ψ vanish at infinity. This shows that then

$$|I(s,\xi_n)| \le \frac{C}{|\xi_n|}\int_0^\infty \left|\frac{d}{dt}(t^{s-1}\psi(t))\right|dt \le \frac{C}{|\xi_n|}.$$

Indeed the larger Re s is, the more integration by parts one can perform and the better decay one obtains as $|\xi| \to \infty$: if Re $s > k$ for some positive integer k, then arguments analogous to above give

$$|I(s,\xi_n)| \le \frac{C}{|\xi_n|^k}. \tag{4}$$

It turns out that one can get a better estimate than the trivial one even if we just have Re $s \in (0,1]$: the idea is that one can in effect perform an integration by parts Re s times even though Re s is not an integer. More precisely, one splits the integral

$$I(s,\xi_n) = \int_0^\varepsilon + \int_\varepsilon^\infty t^{s-1}\psi(t)e^{-2\pi i t\xi_n}dt = A + B,$$

$\varepsilon < 1$. The first term is estimated by

$$|A| \le \int_0^\varepsilon t^{\mathrm{Re}\, s-1}|\psi(t)|dt \le C_{\mathrm{Re}\, s}\varepsilon^{\mathrm{Re}\, s}.$$

The second term can be integrated by parts:

$$\begin{aligned} B &= \int_\varepsilon^\infty t^{s-1}\psi(t)\frac{1}{-2\pi i\xi_n}\frac{d}{dt}\left(e^{-2\pi i t\xi_n}\right)dt \\ &= \frac{1}{2\pi i\xi_n}\left(\varepsilon^{s-1}\psi(\varepsilon)e^{-2\pi i\varepsilon\xi_n} + \int_\varepsilon^\infty \frac{d}{dt}(t^{s-1}\psi(t))e^{-2\pi i t\xi_n}dt\right) \end{aligned}$$

so

$$|B| \leq \frac{C\varepsilon^{\mathrm{Re}\, s-1}}{|\xi_n|} + \frac{C}{|\xi_n|}\int_\varepsilon^\infty \left|\frac{d}{dt}(t^{s-1}\psi(t))\right| dt \leq \frac{C_{\mathrm{Re}\, s}\varepsilon^{\mathrm{Re}\, s-1}}{|\xi_n|}.$$

Hence

$$|I(s,\xi_n)| \leq |A| + |B| \leq C_{\mathrm{Re}\, s}\left(\varepsilon^{\mathrm{Re}\, s} + \frac{\varepsilon^{\mathrm{Re}\, s-1}}{|\xi_n|}\right).$$

Note that we are free to choose ε here. The first term is big when ε is small, while the second term is big when ε is large. One optimizes the expression by choosing ε such that the two terms are of the same order, say by setting

$$\varepsilon^{\mathrm{Re}\, s} = \frac{\varepsilon^{\mathrm{Re}\, s-1}}{|\xi_n|},$$

i.e. $\varepsilon = |\xi_n|^{-1}$. Then

$$|I(s,\xi_n)| \leq C_{\mathrm{Re}\, s}|\xi_n|^{-\mathrm{Re}\, s}, \tag{5}$$

a better decay than the trivial one when $|\xi_n|$ goes to infinity. (5) was proven just now for Re $s \in (0,1]$, but if Re $s \in (k, k+1]$, then one can integrate by parts k times just as we did when we improved from estimate (3) to (4) and that allows one to see that (5) actually holds on the whole right half plane Re $s > 0$. Together with the trivial estimate, we see that

$$|I(s,\xi_n)| \leq C_{\mathrm{Re}\, s}(1+|\xi_n|)^{-\mathrm{Re}\, s} \tag{6}$$

when Re $s > 0$.

Next we estimate $I(s,\xi_n)$ when Re $s \leq 0$. Here we need a different integration by parts to reduce the estimate to the case Re $s \in (0,1]$. By analytic continuation, if Re $s \in (-k, -k+1]$ for some positive integer k, then

$$\frac{1}{\Gamma(s)}I(s,\xi_n) = \frac{(-1)^k}{\Gamma(s+k)}\int_0^\infty t^{s+k-1}\frac{d^k}{dt^k}(\psi(t)e^{-2\pi i t\xi_n})dt$$

where then Re $(s+k) \in (0,1]$. It follows from the proof of (6) that

$$\left|\frac{1}{\Gamma(s)}I(s,\xi_n)\right| \leq \frac{C_{\mathrm{Re}\,(s+k)}}{\Gamma(s+k)}(1+|\xi_n|)^k(1+|\xi_n|)^{-\mathrm{Re}\,(s+k)} \leq C_{\mathrm{Re}\, s}e^{\mathrm{Im}\, s}(1+|\xi_n|)^{-\mathrm{Re}\, s}. \tag{7}$$

Back to L^2 theory

Putting things back together, note that by the assumed decay $\widehat{d\mu} = O(|\xi|^{-\alpha})$ and by the finiteness of the measure $d\mu$, we have

$$|\widehat{d\mu}(\xi)| \leq C(1+|\xi|)^{-\alpha}.$$

From (7), we see that if we were to find a vertical line on the complex s plane on which $\widehat{K_s}(\xi) = \Gamma(s)^{-1}I(s,\xi_n)\widehat{d\mu}(\xi)$ is bounded as a function of ξ, then the best that we can do is to go to the straight line Re $s = -\alpha$; we could not hope to go beyond the left hand side of that. Indeed on this line

$$|\widehat{K_s}(\xi)| \leq C_{\mathrm{Re}\, s} e^{\mathrm{Im}\, s}(1+|\xi_n|)^{-\mathrm{Re}\, s}(1+|\xi|)^{-\alpha} \leq C_\alpha e^{\mathrm{Im}\, s}$$

This proves that T_s is bounded from L^2 to L^2 if Re $s = -\alpha$, with norm at most $C_\alpha e^{\mathrm{Im}\, s}$. Again, this does not quite satisfy the hypothesis of the complex interpolation theorem, but we shall take care of that now.

Conclusion with complex interpolation

Indeed it suffices to modify slightly the definition of T_s to apply the complex interpolation theorem as stated. Let

$$\tilde{T}_s f(x) = e^{s^2} T_s f(x).$$

Then $\tilde{T}_0$ is still the average operator A, but now one takes advantage of that

$$\left|e^{s^2}\right| = e^{(\mathrm{Re}\, s)^2 - (\mathrm{Im}\, s)^2}$$

which decays more rapidly then $e^{-\mathrm{Im}\, s}$ as Im $s \to \infty$. Hence one has now

$$\begin{cases} \|\tilde{T}_{-\alpha+it} f\|_{L^2} \leq C_\alpha \|f\|_{L^2} \\ \|\tilde{T}_{1+it} f\|_{L^\infty} \leq C\|f\|_{L^1} \end{cases}$$

It follows from the complex interpolation theorem that $\tilde{T}_0 = A$ maps $L^{\frac{2\alpha+2}{2\alpha+1}}$ boundedly to $L^{2\alpha+2}$. □

Theorem 3 (Restriction theorem)**.** *Let S, $d\mu$ and α be as in the average theorem. Then the Fourier transform maps $L^p(\mathbb{R}^n)$ to $L^2(S, d\mu)$,*

$$p = \frac{2(\alpha+1)}{\alpha+2}.$$

Proof. Let f be a Schwartz function.

R^*R duality lemma

First,

$$\begin{aligned} \int_S \left|\widehat{f}(\xi)\right|^2 d\mu(\xi) &= \int_S \widehat{f}(\xi)\overline{\widehat{f}(\xi)} d\mu(\xi) \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_S f(y) e^{-2\pi i y\cdot\xi}\overline{f(x)e^{-2\pi i x\cdot\xi}} d\mu(\xi)\, dy\, dx \\ &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} f(y)\widehat{d\mu}(y-x)\, dy\right)\overline{f(x)}\, dx. \end{aligned}$$

This is just another way of saying that if $Rf = \widehat{f}\Big|_S$ then

$$\langle Rf, Rf\rangle_{L^2(S,d\mu)} = \langle R^*Rf, f\rangle_{L^2(\mathbb{R}^n)}$$

with

$$R^*Rf = f * \widehat{d\mu}(-\,\cdot).$$

(Recall that given an $L^2(d\mu)$ function g defined on S, $R^*g(x)$ is the function defined on $\mathbb{R}^n$ such that

$$\langle Rf, g\rangle_{L^2(S,d\mu)} = \langle f, R^*g\rangle_{L^2(\mathbb{R}^n)}$$

say for all Schwartz function f, where again $Rf = \widehat{f}\Big|_S$. But

$$\begin{aligned}
\langle Rf, g\rangle_{L^2(S,d\mu)} &= \int_S \widehat{f}(\xi)\overline{g(\xi)}d\mu(\xi) \\
&= \int_S \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\overline{g(\xi)}dx d\mu(\xi) \\
&= \int_{\mathbb{R}^n} f(x)\overline{\int_S g(\xi)e^{2\pi i x\cdot\xi}d\mu(\xi)}dx \\
&= \left\langle f(x), \int_S g(\xi)e^{2\pi i x\cdot\xi}d\mu(\xi)\right\rangle_{L^2(\mathbb{R}^n)},
\end{aligned}$$

so

$$R^*g(x) = \int_S g(\xi)e^{2\pi i x\cdot\xi}d\mu(\xi)$$

for $g \in L^2(S, d\mu)$. (One can thus think of R^* as an extension operator.) It follows that

$$R^*Rf(x) = \int_S \widehat{f}(\xi)e^{2\pi i x\cdot\xi}d\mu(\xi) = \int_{\mathbb{R}^n} f(y)\widehat{d\mu}(y-x)dy$$

as claimed.)

To prove that R maps L^p to L^2, it suffices to show that R^*R maps $L^p(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$, i.e.

$$\|f * \widehat{d\mu}(-\,\cdot)\|_{L^{p'}(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)},$$

because

$$\|Rf\|^2_{L^2(S,d\mu)} \leq \|R^*Rf\|_{L^{p'}(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}.$$

Embedding into an analytic family of operators

Now

$$R^*Rf(x) = \int_{\mathbb{R}^n} f(x-y)\widehat{d\mu}(-y)dy.$$

Again we would like to embed this into an analytic family of operators. Let K_s be an analytic family of distributions with $K_0 = d\mu$ as in (1) in the proof of the average theorem. Then it is natural to define, motivated by the proof of the average theorem, that

$$T_s f(x) = e^{s^2}\int_{\mathbb{R}^n} f(x-y)\widehat{K_s}(-y)dy.$$

This integral converges for all complex values of s, since we observed that $\widehat{K_s}(y)$ is a function that grows at most polynomially in y for each fixed complex value of s, and f is Schwartz.

L^2 theory

Observe that

$$\widehat{T_s f}(\xi) = e^{s^2}\widehat{f}(\xi)K_s(-\xi)$$

and recall that

$$\left|e^{s^2}K_s(-\xi)\right| \leq C$$

if Re $s = 1$. Hence when Re $s = 1$, T_s maps L^2 to L^2 with norm at most C.

L^1 theory

Next recall that

$$\left|e^{s^2}\widehat{K_s}(\xi)\right| \leq C_\alpha$$

if Re $s = -\alpha$. Hence when Re $s = -\alpha$, T_s maps L^1 to L^∞ with norm at most C_α.

Conclusion with complex interpolation

It follows from the complex interpolation theorem now that $R^*R = T_0$ maps $L^p(\mathbb{R}^n)$ boundedly to $L^{p'}(\mathbb{R}^n)$, $p = \frac{2(\alpha+1)}{\alpha+2}$. Hence the Fourier transform extends to a map from $L^p(\mathbb{R}^n)$ to $L^2(S, d\mu)$. □

Corollary 1. *Let S, $d\mu$, α be as in the restriction theorem. Then the extension operator*

$$R^*g(x) = \int_S g(\xi)e^{2\pi i x\cdot\xi}d\mu(\xi)$$

maps $L^2(S, d\mu)$ boundedly to $L^q(\mathbb{R}^n)$, with

$$q = \frac{2(\alpha+1)}{\alpha}.$$

Proof. Note that the q in this corollary is the conjugate exponent to p in the restriction theorem. We have seen R maps $L^p(\mathbb{R}^n)$ to $L^2(S, d\mu)$, $p = \frac{2(\alpha+1)}{\alpha+2}$. As a result,

$$\begin{aligned}\left|\langle f, R^*g\rangle_{L^2(\mathbb{R}^n)}\right| &= \left|\langle Rf, g\rangle_{L^2(S,d\mu)}\right| \\ &\leq \|Rf\|_{L^2(S,d\mu)}\|g\|_{L^2(S,d\mu)} \\ &\leq C\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^2(S,d\mu)}\end{aligned}$$

so by duality

$$\|R^*g\|_{L^q(\mathbb{R}^n)} \leq C\|g\|_{L^2(S,d\mu)},$$

q being the conjugate exponent to p, i.e. $q = \frac{2(\alpha+1)}{\alpha}$. □

We shall now give some applications of the above theorems. Here we shall be dealing with non-compact submanifolds of an Euclidean space, and the measure involved will not be the one that is induced from the Lebesgue measure of the underlying Euclidean space. The proofs illustrate how one handles such non-compactness by introducing cut-off functions.

Corollary 2 (Strichartz estimate for the linear Schrodinger equation). *Let $f(x)$ be a Schwartz function on $\mathbb{R}^n$. If*

$$u(x,t) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i (x\cdot\xi + t|\xi|^2)} d\xi$$

so that u solves the linear Schrodinger equation

$$\left(2\pi i \frac{\partial}{\partial t} - \Delta_x\right) u(x,t) = 0$$

with initial value

$$u(x,0) = f(x),$$

then

$$\|u\|_{L^q(dxdt)} \leq C\|f\|_{L^2(dx)}$$

with

$$q = \frac{2(n+2)}{n}.$$

Proof. The crucial submanifold here is the hypersurface

$$S = \{(\xi,\tau) \in \mathbb{R}^{n+1} \colon \tau = |\xi|^2\}.$$

The induced Lebesgue measure on the hypersurface is $d\sigma = \sqrt{1+4|\xi|^2} d\xi$, but the essential property for us would be its behaviour near the origin. We would like to apply the average theorem and the restriction theorem, but our surface-carried measure $d\sigma$ does not have compact support. Therefore we need to introduce a cut-off function and let the support go to infinity. The non-isotropic dilations,

$$(x,t) \mapsto (\lambda x, \lambda^2 t)$$

that preserve the hypersurface, will also play a role.

Let $\eta \in C_c^\infty(\mathbb{R}^n)$ with $\eta(0) = 1$, and let $d\mu = \eta(\xi) d\sigma = \eta(\xi)\sqrt{1+4|\xi|^2} d\xi$. Then since S has nowhere vanishing Gaussian curvature, $\widehat{d\mu}(x,t)$ decays like

$$\widehat{d\mu}(x,t) = O(|(x,t)|^{-\frac{n}{2}}),$$

so we can apply Corollary 1 with $\alpha = n/2$.

The corollary to the restriction theorem says that if

$$Tf(x,t) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i (x\cdot\xi + t|\xi|^2)} \eta(\xi)\sqrt{1+4|\xi|^2} d\xi$$

then

$$\|Tf\|_{L^q(dxdt)} \leq C\|\widehat{f}\|_{L^2(d\xi)} = C\|f\|_{L^2(dx)},$$

where $q = \frac{2(n+2)}{n}$. Now pointwisely

$$\begin{aligned} u(x,t) &= \lim_{\lambda\to+\infty} \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i(x\cdot\xi + t|\xi|^2)} \eta\left(\frac{\xi}{\lambda}\right) \sqrt{1 + 4\left|\frac{\xi}{\lambda}\right|^2} d\xi \\ &= \lim_{\lambda\to+\infty} \int_{\mathbb{R}^n} \lambda^n \widehat{f}(\lambda\xi) e^{2\pi i(\lambda x\cdot\xi + \lambda^2 t|\xi|^2)} \eta(\xi) \sqrt{1+4|\xi|^2} d\xi \\ &= \lim_{\lambda\to+\infty} T(f_\lambda)(\lambda x, \lambda^2 t) \end{aligned}$$

where $f_\lambda(x) = f(\lambda^{-1}x)$. Hence for $q = \frac{2(n+2)}{n}$,

$$\begin{aligned} \|u\|_{L^q(dxdt)} &\le \liminf_{\lambda\to+\infty} \|T(f_\lambda)(\lambda x, \lambda^2 t)\|_{L^q(dxdt)} && \text{(Fatou's lemma)} \\ &= \liminf_{\lambda\to+\infty} \lambda^{-\frac{n+2}{q}} \|T(f_\lambda)\|_{L^q(dxdt)} && \text{(Scale invariance)} \\ &\le \liminf_{\lambda\to+\infty} C\lambda^{-\frac{n+2}{q}} \|f_\lambda\|_{L^2(dx)} && \text{(Restriction theorem)} \\ &= \liminf_{\lambda\to+\infty} C\lambda^{-\frac{n+2}{q}} \lambda^{\frac{n}{2}} \|f\|_{L^2(dx)} && \text{(Scale invariance)} \\ &= C\|f\|_{L^2(dx)} \end{aligned}$$

since

$$-\frac{n+2}{q} + \frac{n}{2} = 0$$

for this value of q. Hence we are done. □

Corollary 3 (Averaging along a paraboloid)**.** *Let $f(x)$ be a Schwartz function on $\mathbb{R}^n$. If*

$$Af(x) = \int_{\mathbb{R}^{n-1}} f(x' - y', x_n - |y'|^2) dy'$$

then

$$\|Af\|_{L^{p'}(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}$$

with

$$p = \frac{n+1}{n}.$$

Proof. Again let $d\mu = \eta(y')\sqrt{1+4|y'|^2}dy'$ be a measure carried on the paraboloid $\{y_n = |y'|^2\}$, where $\eta \in C_c^\infty(\mathbb{R}^{n-1})$. Then $\widehat{d\mu}(\xi) = O(|\xi|^{-\frac{n-1}{2}})$ because S has nowhere vanishing Gaussian curvature. Hence the average theorem says that if

$$Tf(x) = \int_{\mathbb{R}^{n-1}} f(x' - y', x_n - |y'|^2)\eta(y')\sqrt{1+4|y'|^2}dy',$$

then

$$\|Tf\|_{L^{p'}(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}$$

with

$$p = \frac{n+1}{n}.$$

Now pointwisely

$$\begin{aligned} Af(x) &= \int_{\mathbb{R}^{n-1}} f(x' - y', x_n - |y'|^2) dy' \\ &= \lim_{\lambda \to +\infty} \int_{\mathbb{R}^{n-1}} f(x' - y', x_n - |y'|^2) \eta\left(\frac{y'}{\lambda}\right) \sqrt{1 + 4\left|\frac{y'}{\lambda}\right|^2} dy' \end{aligned}$$

according to the dominated convergence theorem. However,

$$\begin{aligned} &\int_{\mathbb{R}^{n-1}} f(x' - y', x_n - |y'|^2) \eta\left(\frac{y'}{\lambda}\right) \sqrt{1 + 4\left|\frac{y'}{\lambda}\right|^2} dy' \\ =&\lambda^{n-1} \int_{\mathbb{R}^{n-1}} f(x' - \lambda y', x_n - \lambda^2 |y'|^2) \eta(y') \sqrt{1 + 4|y'|^2} dy' \\ =&\lambda^{n-1} T(f_\lambda)(\lambda^{-1} x', \lambda^{-2} x_n) \end{aligned}$$

where

$$f_\lambda(x) = f(\lambda x', \lambda^2 x_n)$$

is the dilation of f by the relevant non-isotropic dilation. Hence

$$\begin{aligned} \|Af\|_{L^{p'}(\mathbb{R}^n)} &\leq \liminf_{\lambda \to +\infty} \lambda^{n-1} \|T(f_\lambda)(\lambda^{-1} x', \lambda^{-2} x_n)\|_{L^{p'}(\mathbb{R}^n)} && \text{(Fatou's lemma)} \\ &= \liminf_{\lambda \to +\infty} \lambda^{n-1} \lambda^{\frac{n+1}{p'}} \|T(f_\lambda)\|_{L^p(\mathbb{R}^n)} && \text{(Scale invariance)} \\ &\leq \liminf_{\lambda \to +\infty} C \lambda^{n-1} \lambda^{\frac{n+1}{p'}} \|f_\lambda\|_{L^p(\mathbb{R}^n)} && \text{(Average theorem)} \\ &= \liminf_{\lambda \to +\infty} C \lambda^{n-1} \lambda^{\frac{n+1}{p'}} \lambda^{-\frac{n+1}{p}} \|f\|_{L^p(\mathbb{R}^n)} && \text{(Scale invariance)} \\ &= C \|f\|_{L^p(\mathbb{R}^n)} \end{aligned}$$

since

$$n - 1 + \frac{n+1}{p'} - \frac{n+1}{p} = 0$$

when $p = \frac{n+1}{n}$. This completes the proof. □

Corollary 4 (Strichartz estimate for linearized KdV equation)**.** *Let $f(x)$ be a Schwartz function on $\mathbb{R}$. If*

$$u(x,t) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i (x\xi + t\xi^3)} d\xi$$

so that u solves the linearized KdV equation

$$\left(4\pi^2 \frac{\partial}{\partial t} - \frac{\partial^3}{\partial^3 x}\right) u(x,t) = 0$$

with initial value

$$u(x,0) = f(x),$$

then

$$\|u\|_{L^8(dxdt)} \leq C \|f\|_{L^2(dx)}.$$

Proof. The proof is the same as that of Corollary 2, except now that the relevant submanifold is the curve $S = \{(\xi, \xi^3) \colon \xi \in \mathbb{R}\}$ in $\mathbb{R}^2$ and the curvature of S vanishes at the origin. The best possible decay for $\widehat{d\mu}$ is now

$$\widehat{d\mu}(\xi) = O(|\xi|^{-\frac{1}{3}}),$$

so if we apply Corollary 1 as in the proof of Corollary 2 with $\alpha = 1/3$ we obtain

$$\|u\|_{L^8(dxdt)} \leq C\|f\|_{L^2(dx)}.$$

□

Corollary 5 (Averaging along a curve of finite type)**.** *Let $f(x)$ be a Schwartz function on $\mathbb{R}^2$. If*

$$Af(x) = \int_{\mathbb{R}} f(x_1 - t, x_2 - t^k)dt$$

then

$$\|Af\|_{L^{p'}(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}$$

with

$$p = \frac{2k+2}{k+2}.$$

Proof. Again the proof is the same as that of Corollary 3, except now that the relevant submanifold is the curve $S = \{(t, t^k) \colon t \in \mathbb{R}\}$ in $\mathbb{R}^2$. The curvature of S vanishes at the origin. However, since it is a curve of finite type k, we still have a certain decay for $\widehat{d\mu}$; indeed

$$\widehat{d\mu}(\xi) = O(|\xi|^{-\frac{1}{k}}),$$

so if we apply Theorem 1 with $\alpha = 1/k$ as in the proof of Corollary 3 we obtain

$$\|Af\|_{L^{p'}(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}$$

with

$$p = \frac{2k+2}{k+2}.$$

□