Fourier Analysis and Oscillatory Integrals

Problems from the course by Elias Stein

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These solutions were put together by the participants of the 2006 Princeton summer school in analysis and geometry. Thanks to Kiril Datchev, the original version is available from his webpage, http://math.berkeley.edu/∼datchev. The participants of the same program in 2007 has contributed some alternative solutions and additional remarks.

1. *Suppose* $f \in L^1(\mathbb{R}^d)$. *Show that* $\hat{f}$ *is continuous and that* $\hat{f} \to 0$ *as* $\xi \to \infty$.

   (a) Continuity: Suppose $\xi_n \to \xi$. We want $\hat{f}(\xi_n) \to \hat{f}(\xi)$, or

$$\int f(x)e^{-2\pi i x\cdot\xi_n}dx \to \int f(x)e^{-2\pi i x\cdot\xi}dx$$

   For each x, we have

$$f(x)e^{-2\pi i x\cdot\xi_n} \to f(x)e^{-2\pi i x\cdot\xi}$$

   and moreover, since $|e^{-2\pi i x\cdot\xi}| = 1$, it follows that,

$$|f(x)e^{-2\pi i x\cdot\xi}| \le |f(x)| \in L^1.$$

   We can therefore apply the Dominated Convergence Theorem to yield the desired conclusion.

   Alternative solution. To prove continuity of $\hat{f}$, it is natural to ask a harder question and try obtaining some quantitative bounds for the decay of $\hat{f}(\xi+h) - \hat{f}(\xi)$ as $h \to 0$. If f is Schwartz, then $\hat{f}$ is also Schwartz, so

$$\hat{f}(\xi+h) - \hat{f}(\xi) = O(|h|)$$

   as $h \to 0$ for such f. In general, such an inequality cannot be expected for a general L^1 function f (why?); however, given a general L^1 function f, one could approximate it by a Schwartz function g in the L^1 norm, i.e. one could pick a Schwartz function g such that

$$\|f-g\|_{L^1} < \varepsilon.$$

Then

$$
\begin{aligned}
|\hat{f}(\xi+h)-\hat{f}(\xi)| &\le |\hat{f}(\xi+h)-\hat{g}(\xi+h)|+|\hat{g}(\xi+h)-\hat{g}(\xi)|+|\hat{g}(\xi)-\hat{f}(\xi)| \\
&\le 2\|f-g\|_{L^1}+O(|h|) \\
&\le 2\varepsilon+O(|h|),
\end{aligned}
$$

and letting $h \to 0$ and $\varepsilon \to 0$, we get $\hat{f}(\xi+h) \to \hat{f}(\xi)$ as $h \to 0$.

Remark. It was observed in the problem session that this second proof actually proves the uniform continuity of $\hat{f}$ for a general L^1 function f. Also, one might use instead of the Schwartz functions the L^1 functions with compact support; such are still dense in L^1 and $|\hat{f}(\xi+h)-\hat{f}(\xi)| \le \|f(x)(e^{2\pi i h\cdot x}-1)\|_{L^1} \le R|h|\|f\|_{L^1} = O(|h|)$ for such f, where R is the size of the support of f.

(b) That

$$\lim_{\xi\to\infty}\hat{f}(\xi)=0:$$

Explicit calculation shows that this is true if f is the characteristic function of an n-dimensional interval. For example, in the one-dimensional case, if $f=\chi_{[a,b]}$, then

$$\hat{f}(\xi)=\int_{-\infty}^{\infty} f(x)e^{-2\pi i\xi x}dx=\int_a^b e^{-2\pi i\xi x}dx=\int_a^b \cos(-2\pi\xi x)dx+i\int_a^b \sin(-2\pi\xi x)dx$$

which equals

$$\frac{1}{2\pi\xi}((\sin 2\pi b\xi-\sin 2\pi a\xi)+i(\cos 2\pi b\xi-\cos 2\pi a\xi)),$$

a quantity that approaches 0 as ξ approaches ∞. (By the way, this is the most basic instance of a van der Corput's estimates: the proof shows $\left|\int_a^b e^{i\xi x}dx\right| \le 4/|\xi|$ independent of a and b, which is the van der Corput's estimate for the case where the phase function $\phi(x)=x$.) It follows that the same conclusion holds for finite linear combinations of such characteristic functions. Now such "simple" functions are dense in L^1, so we argue as follows: we want to show that

$$|\lim_{\xi\to\infty}\hat{f}(\xi)|<\epsilon$$

for arbitrary $\epsilon>0$. If $\{g_n\}$ is a sequence of simple functions converging to f in the L^1-norm, then by what we have just seen,

$$|\lim_{\xi\to\infty}\hat{f}(\xi)|=|\lim_{\xi\to\infty}\hat{f}(\xi)-\lim_{\xi\to\infty}\hat{g_n}(\xi)|$$

for each n. But

$$|\lim_{\xi\to\infty}\hat{f}(\xi)-\lim_{\xi\to\infty}\hat{g_n}(\xi)|=\lim_{\xi\to\infty}|\hat{f}(\xi)-\hat{g_n}(\xi)|,$$

which, by the linearity of the Fourier transform, is equal to

$$\lim_{\xi\to\infty}|(f-g_n)^\wedge(\xi)|.$$

By continuity of $(f-g_n)^\wedge$ (see part (a) above), we have

$$\lim_{\xi\to\infty}|(f-g_n)^\wedge(\xi)| \leq \sup|(f-g_n)^\wedge(\xi)| = \|(f-g_n)^\wedge\|_\infty \leq \|(f-g_n)\|_1 < \epsilon$$

for n sufficiently large, as was to be shown.

Alternative solution. One could also approximate using compactly supported smooth functions, or Schwartz functions.

2. *Suppose* $f \in L^1(\mathbb{R}^d)$. *Prove*

   (a) $\int |f(x+h)-f(x)|dx \to 0$ *as* $|h| \to 0$. Let $\varepsilon > 0$ be given. Choose f_ε continuous and compactly supported such that $||f-f_\varepsilon||_1 < \varepsilon/3$ ($||\cdot||_1$ denotes the L^1 norm). Now

   $$||f(x+h)-f(x)||_1 \leq ||f(x+h)-f_\varepsilon(x+h)||_1+||f_\varepsilon(x+h)-f_\varepsilon(x)||_1+||f_\varepsilon(x+h)-f(x)||_1$$

   The first and last terms are already bounded by $\varepsilon/3$. We need only choose h such that

   $$|f_\varepsilon(x+h)-f_\varepsilon(x)| < \frac{\varepsilon}{3M}$$

   where M is the measure of the support of f_ε. But this is possible because a compactly supported continuous function is uniformly continuous. Since this is possible for any $\varepsilon > 0$, the proof is complete.

   (b) *If* $\int |f(x+h)-f(x)|dx \leq A|h|^\alpha$ *as* $|h| \to 0$, *then* $\hat{f}(\xi) = O(|\xi|^{-\alpha})$, $\xi \to \infty$.
   The condition on the Fourier Transform is deduced as follows:

   $$A|h|^\alpha \geq \int |f(x+h)-f(x)|dx \geq |\int e^{-2\pi i x\xi}(f(x+h)-f(x))dx| = |\hat{f}(\xi)(e^{2\pi i\xi h}-1)|$$

   This means that

   $$|\hat{f}(\xi)||\frac{e^{2\pi i\xi h}-1}{|h|^\alpha}| \leq A \tag{1}$$

   For sufficiently small $|h|$ this is true for all ξ; suppose that the threshold is $|h| < \varepsilon$. We now choose h such that $e^{2\pi i\xi h} \neq 1$ and write

   $$|\hat{f}(\xi)||\xi|^\alpha \leq A\frac{(|h||\xi|)^\alpha}{|e^{2\pi i\xi h}-1|}$$

   We will now show that when $|\xi| > 1/\varepsilon$, we can find h with $|h| < \varepsilon$ such that

   $$\frac{(|h||\xi|)^\alpha}{|e^{2\pi i\xi h}-1|} \leq C$$

   To do this, let $h = \frac{\xi}{2|\xi|^2}$. This makes the numerator $(1/2)^\alpha$ and the denominator 2. This completes the proof.

   Remark. The question basically asks one to deduce some sort of decay for the Fourier transform $\hat{f}$ given some smoothness of f. If we knew $\nabla f \in L^1$, then we can integrate by parts to obtain $|\hat{f}(\xi)| = |\int f(x)\frac{1}{-2\pi\xi_j}\frac{\partial}{\partial x_j}e^{-2\pi i x\xi}dx| \leq c\|\nabla f\|_{L^1}/|\xi_j|$. Now we do not have such strong pointwise differentiability condition; we just have a condition involving the difference quotients of f. But the idea is that we can still perform some discrete integration by parts, a version that works for difference quotients, to obtain the same kind of bounds. Note that

$$\int (f(x+h)-f(x))g(x)dx = \int f(x)(g(x-h)-g(x))dx,$$

if each term in the integrand is integrable. Apply this to our f and take $g(x) = e^{-2\pi i x\xi}$, we get

$$\int (f(x+h)-f(x))e^{-2\pi i x\xi}dx = \int f(x)(e^{-2\pi i(x-h)\xi} - e^{-2\pi i x\xi})dx = \hat{f}(\xi)(e^{2\pi i h\xi}-1),$$

as we had above, and noting that we are free to choose h so that $e^{2\pi i h\xi}-1$ is bounded away from 0, we are done.

(c) *If $\alpha > 1$ in (b), then $f = 0$ a.e.*
From (1) above we see that for ξ fixed

$$|\hat{f}(\xi)||(2\pi i\xi h|h|^{-\alpha} + O(|h|^{2-\alpha}))| \leq A$$

If $\alpha > 1$ this is only possible if $\hat{f}(\xi) = 0$. This implies $f = 0$ a.e.
Alternative solution. As was suggested in the problem session, one could prove, without Fourier analysis, that if $\int |f(x+h)-f(x)|dx \leq A|h|^\alpha$ as $|h| \to 0$ for some $\alpha > 1$, then the distributional derivatives of f are all zero, and thus f is constant; being in L^1 we must have $f = 0$ a.e. To see that all distributional derivatives of f are zero, note that if $\phi \in C_c^\infty$, then

$$\begin{aligned}\left|\int f(x)\frac{\phi(x)-\phi(x-h)}{|h|}dx\right| &= \left|-\int \frac{f(x+h)-f(x)}{|h|}\phi(x)dx\right| \\ &\leq C\int \frac{|f(x+h)-f(x)|}{|h|}dx \\ &= O(|h|^{\alpha-1}) \to 0\end{aligned}$$

as $h \to 0$. However, the left hand side converges to $\int f(x)\frac{\partial}{\partial x_j}\phi(x)dx$ as $h \to 0$ along the x_j direction; this is a consequence of the dominated convergence theorem, since $f \in L^1$ and $\phi \in C_c^\infty$. This shows that all distributional derivatives of f are zero.

3. *The result in Problem 1 cannot essentially be improved. Prove that there is an $F \in L^1(\mathbb{R}^d)$ such that for every $\epsilon > 0$, $\hat{F}(\xi) \neq O(|\xi|^{-\epsilon})$ as $\xi \to \infty$*
In $\mathbb{R}^d$, Let $F(x) = \sum_{n=2}^\infty \frac{1}{n(\log n)^2} n^d e^{-\pi|x|^2 n^2}$. Since each $n^d e^{-\pi|x|^2n^2}$ integrates to 1 over $\mathbb{R}^d$ (substitute nx for x in the Gaussian integral), and since $\sum_{n=2}^\infty \frac{1}{n(\log n)^2}$ converges (compare the sum to the integral of $-\frac{1}{x(\log x)^2}$ which is the derivative of $\frac{1}{\log x}$), we see by the monotone convergence theorem that F is in $L^1(\mathbb{R}^d)$. The Fourier transform of $n^d e^{-\pi|x|^2n^2}$ is $e^{-\pi|x|^2/n^2}$

(see the Appendix). Now by the dominated convergence theorem, the fourier transform of $F(x)$ is $G(x) = \sum_{n=2}^{\infty} \frac{1}{n(\log n)^2} e^{-\pi|x|^2/n^2}$. Then for any $x \in \mathbb{R}^d$ with $|x| \geq 2$, we have

$$G(x) = \sum_{n=2}^{\infty} \frac{1}{n(\log n)^2} e^{-\pi|x|^2/n^2} \geq \sum_{n \geq |x|} \frac{1}{n(\log n)^2} e^{-\pi|x|^2/n^2} \geq e^{-\pi} \sum_{n \geq |x|} \frac{1}{n(\log n)^2} \geq e^{-\pi} \frac{1}{\log |x|},$$

where the last inequality comes from comparing the sum $\sum_{n \geq |x|} \frac{1}{n(\log n)^2}$ to the integral $\int_{y \geq |x|} \frac{1}{y(\log y)^2} dy$. It is easy to show using L'Hospital's rule that $\frac{1}{\log |x|}$ is not $O(|x|^{-\epsilon})$ for any $\epsilon > 0$, so the same holds for $G(x)$.

Alternative solution. One good way to construct an L^1 function is to take a sequence of L^1 functions f_j, each with norm 1, and let $F = \sum a_j f_j$, where $\sum |a_j| < \infty$. Indeed then each $\hat{f}_j$ has L^∞ norm bounded by 1, and we can make them narrow non-negative bumps around any point that has height 1; then each $a_j \hat{f}_j$ contributes a height a_j around a prescribed point ξ_j, and if ξ_j goes to infinity fast enough, then the decay of $\hat{F}$ at infinity is worse than any $O(|\xi|^{-\alpha})$. (This last idea was due to the participants of the summer school 2007.) To be more precise, let g be a Schwartz function whose Fourier transform is non-negative, compactly supported in the unit ball and has value 1 at the origin, and ξ_j be a sequence of points to be determined, but which goes to infinity very rapidly as $j \to \infty$. Then $\hat{g}(\xi - \xi_j)$ is supported near ξ_j. Let f_j be the L^1 function whose Fourier transform is $\hat{g}(\xi - \xi_j)$; each f_j has the same L^1 norm, namely the L^1 norm of g. Now let a_j be, say, j^{-2}. Then

$$\hat{F}(\xi_k) = \sum a_j \hat{f}_j(\xi_k) \geq a_k \hat{f}_k(\xi_k) = a_k = k^{-2}.$$

Hence if ξ_k goes to infinity rapidly enough, say $\xi_k = k^k$, then $\hat{F}(\xi)$ is not $O(|\xi|^{-\alpha})$ for any $\alpha > 0$.

Remark. Even in the original solution, the Gaussian function is not essential. If g is any L^1 function whose Fourier transform is non-negative and $\hat{g}(\xi) \geq 1$ for all $|\xi| < 1$, then letting $f_j \in L^1$ be such that $\hat{f}_j(\xi) = \hat{g}(\xi/j)$, we have all $\|f_j\|_{L^1}$ being equal, and $F = \sum a_j f_j$ would be in L^1 as long as each $a_j \geq 0$ and $\sum a_j < \infty$. Now $\hat{f}_j$ is becoming fatter and fatter, and thus $\hat{F}(m) \geq \sum_{j=m}^{\infty} a_j \hat{f}_j(m) = \sum a_j \hat{g}(m/j) \geq \sum_{j=m}^{\infty} a_j$. Thus if we could find a non-negative sequence a_j for which $\sum_{j=1}^{\infty} a_j < \infty$ and $\sum_{j=m}^{\infty} a_j$ decays slower than any negative power of m, say $\sum_{j=m}^{\infty} a_j \geq 1/\log m$, then we are done. But this could easily be done, by taking a telescoping series; indeed if we set $a_j = \frac{1}{\log j} - \frac{1}{\log(j+1)}$ for large j, then the conditions are satisfied, and F as such would be in L^1 and yet having Fourier transform that decays slower than any negative power of $|\xi|$. A trickier choice of a_j would be $a_j = \frac{1}{j(\log j)^2}$.

4. (a) *Suppose $f \in L^2_k(\mathbb{R}^d)$, and $k > d/2$. Show that f can be corrected on a set of measure zero to become continuous.*

Let $f \in L^2_k(\mathbb{R}^d)$ with $k > d/2$. Then, by definition, $\hat{f}(\xi)(1 + |\xi|^2)^{k/2} \in L^2(\mathbb{R}^d)$. By the Schwarz-Cauchy inequality it follows that

$$\hat{f}(\xi) = (\hat{f}(\xi)(1 + |\xi|^2)^{k/2}) \cdot (1 + |\xi|^2)^{-k/2} \in L^1(\mathbb{R}^d)$$

Using Problem 1 above, it follows that $\hat{\hat{f}}$ lies in the same equivalence class of $L^\infty(\mathbb{R}^d)$ as some continuous function, so $\hat{\hat{f}}$ (and hence also f), can be modified on a set of measure zero to become continuous.

Remark. If $k > d/p$, then every $f \in L^p_k(\mathbb{R}^d)$ can be modified on a set of measure zero to become continuous: this is the General Sobolev Theorem, proved on pp. 270-271 of *Partial Differential Equations* by L.C. Evans. More on Sobolev spaces can be found in Chapter 6.5 of Stein's *Harmonic Analysis.*

(b) *Give an example of $f \in L^2_1(\mathbb{R}^2)$ which cannot be corrected to be continuous.*

Let $f(x) = \log^\alpha(1/|x|))\chi(x)$, where $\chi \in C_0^\infty(\mathbb{R}^2)$, $\operatorname{supp}(\chi) \subset B(0,2)$; $\chi \equiv 1$ near 0, and $\alpha \in (0, 1/2)$. Since $\lim_{|x|\to 0} f(x) = +\infty$, f can't be modified on a set of measure zero to be continuous.

Claim. $f \in L^2_1(\mathbb{R}^2)$

Proof. We use polar coordinates on $\mathbb{R}^2$

$$\int_{\mathbb{R}^2} |f(x)|^2 dx \le C \int_0^\infty \log^{2\alpha}(1/r) r dr$$

We will show

$$\lim_{r\to 0^+} \log^{2\alpha}(1/r) r = 0 \tag{2}$$

from which it will follow that the the latter integral is finite. But this limit is the same as

$$\lim_{x\to\infty} \frac{\log^\alpha x}{x} = \lim_{x\to\infty} \frac{\alpha \log^{\alpha-1} x(1/x)}{1} = 0$$

where we have used L'Hospital's rule. Now (2) holds.

Let $j \in \{1, 2\}$. Then, near 0, $\partial_j f(x) = \partial_j \log^\alpha(1/|x|) = -\alpha \log^{\alpha-1}(1/|x|)\frac{x_j}{2|x|^2}$ So

$$|\partial_j f(x)| \le C \log^{\alpha-1}(1/|x|)\frac{1}{|x|}$$

for $x \neq 0$ sufficiently small. Using polar coordinates, it will follow that $\partial_j f \in L^2(\mathbb{R}^2)$ if we manage to show that

$$\int_0^{1/2} \log^{2(\alpha-1)}(1/r)\frac{rdr}{r^2} = \int_0^{1/2} \log^{2(\alpha-1)}(1/r)\frac{dr}{r} < +\infty \tag{3}$$

Observe that $\frac{d}{dr}\log^{2\alpha-1}(1/r) = -(2\alpha - 1)\log^{2(\alpha-1)}(1/r)/r$ for $0 < r < 1/2$. Also $2\alpha - 1 < 0$ by construction, hence

$$\lim_{\varepsilon\to 0^+} \log^{2\alpha-1}(1/r)\Big|_{r=\varepsilon}^{r=1/2} = \log^{2\alpha-1} 2$$

So (3) immediately follows. ∎

Thus it follows that $f \in L_1^2(\mathbb{R}^2)$.

Alternative solution. $x \mapsto \log\log|x|$ is in L_1^2 near the origin in $\mathbb{R}^2$, but cannot be modified to be continuous at the origin.

5. (a) *Show that*

$$J_{-\frac{1}{2}}(r) = \lim_{k\to-\frac{1}{2}} J_k(r) = \sqrt{\frac{2}{\pi}}\frac{\cos r}{r^{\frac{1}{2}}}$$

and

$$J_{\frac{1}{2}}(r) = \sqrt{\frac{2}{\pi}}\frac{\sin r}{r^{\frac{1}{2}}}.$$

Proof:

We take the Bessel functions $J_k(r)$ to be defined as

$$J_k(r) = \frac{\left(\frac{r}{2}\right)^k}{\Gamma(k+\frac{1}{2})\sqrt{\pi}}\int_{-1}^{1} e^{irs}(1-s^2)^{k-\frac{1}{2}}ds,$$

where $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}$ is the usual Γ function and satisfies $\Gamma(1) = 0! = 1$.

For the second identity, we calculate

$$\begin{aligned}\int_{-1}^{1} e^{irs}(1-s^2)^{\frac{1}{2}-\frac{1}{2}}ds &= \int_{-1}^{1}(\cos(rs) + i\sin(rs))ds \\ &= \int_{-1}^{1}\cos(rs)ds + \int_{-1}^{1} i\sin(rs)ds \\ &= 2\int_0^1 \cos(rs)ds + 0 \\ &= 2\frac{\sin(r)}{r},\end{aligned}$$

which implies

$$\begin{aligned}J_{\frac{1}{2}}(r) &= \frac{\left(\frac{r}{2}\right)^{\frac{1}{2}}}{\Gamma(1)\sqrt{\pi}}\int_{-1}^{1} e^{irs}(1-s^2)^{\frac{1}{2}-\frac{1}{2}}ds \\ &= \left(r^{\frac{1}{2}}\sqrt{\frac{1}{2\pi}}\right)\left(2\frac{\sin(r)}{r}\right) \\ &= \sqrt{\frac{2}{\pi}}\frac{\sin(r)}{r^{\frac{1}{2}}}\end{aligned}$$

as wanted.

For the first identity, assume $k > -\frac{1}{2}$. In the above definition of the Bessel functions $J_k(r)$, we have that as k approaches $-\frac{1}{2}$ from the right, $\Gamma(k+\frac{1}{2})$ approaches $+\infty$ and

the $(1-s^2)^{k-\frac{1}{2}}$ term makes the integral blow up near $s=1$. So, we try to get these terms to balance using integration by parts.
We use the formula $\Gamma(k+\frac{3}{2}) = (k+\frac{1}{2})\Gamma(k+\frac{1}{2})$ (which is valid for $k > -\frac{1}{2}$), and as before we write $e^{irs} = \cos(rs) + i\sin(rs)$ and break the integral into two. The first is the integral of the even function $\cos(rs)(1-s^2)^{k-\frac{1}{2}}$, so it is twice the integral on $[0,1]$ of the same thing. The second is the integral on $[-1,1]$ of the odd function $\sin(rs)(1-s^2)^{k-\frac{1}{2}}$, and is thus 0. We are left with

$$J_k(r) = \frac{\left(\frac{r}{2}\right)^k}{\Gamma(k+\frac{3}{2})\sqrt{\pi}} 2(k+\frac{1}{2}) \int_0^1 \cos(rs)(1-s^2)^{k-\frac{1}{2}} ds.$$

Since

$$\lim_{k\to-\frac{1}{2}} \frac{\left(\frac{r}{2}\right)^k}{\Gamma(k+\frac{3}{2})\sqrt{\pi}} = \sqrt{\frac{2}{\pi}} r^{-\frac{1}{2}},$$

it suffices to show that

$$\lim_{k\to-\frac{1}{2}} 2(k+\frac{1}{2}) \int_0^1 \cos(rs)(1-s^2)^{k-\frac{1}{2}} ds = \cos(r).$$

On any interval $[0, 1-\epsilon]$, $\cos(rs)(1-s^2)^{k-\frac{1}{2}}$ is bounded uniformly in k, so we see that

$$\lim_{k\to-\frac{1}{2}} 2(k+\frac{1}{2}) \int_0^{1-\epsilon} \cos(rs)(1-s^2)^{k-\frac{1}{2}} ds = 2(0)\cdot \int_0^{1-\epsilon} \cos(rs)(1-s^2)^{-\frac{1}{2}-\frac{1}{2}} ds = 0.$$

For $[1-\epsilon, 1]$ we integrate by parts to get

$$\begin{aligned}
&2(k+\frac{1}{2}) \int_{1-\epsilon}^1 \cos(rs)(1-s^2)^{k-\frac{1}{2}} ds = \\
&= \int_{1-\epsilon}^1 \left(\frac{\cos(rs)}{-s}\right)\left((k+\frac{1}{2})(1-s^2)^{k-\frac{1}{2}}(-2s)\right) ds \\
&= \left[\left(\frac{\cos(rs)}{-2s}\right)\left((1-s^2)^{k+\frac{1}{2}}\right)\right]_{1-\epsilon}^1 - \int_{1-\epsilon}^1 \left(\frac{\cos(rs)}{-s}\right)'(1-s^2)^{k+\frac{1}{2}} ds \\
&= \frac{\cos(r(1-\epsilon))}{1-\epsilon}(2\epsilon-\epsilon^2)^{k+\frac{1}{2}} - \int_{1-\epsilon}^1 \left(\frac{\cos(rs)}{-s}\right)'(1-s^2)^{k+\frac{1}{2}} ds,
\end{aligned}$$

which in the limit as $k \to -\frac{1}{2}$ becomes

$$\frac{\cos(r(1-\epsilon))}{1-\epsilon} - \int_{1-\epsilon}^1 \left(\frac{\cos(rs)}{-s}\right)' ds.$$

Letting $\epsilon \to 0$, the integral term goes to 0 while the boundary term converges to $\cos(r)$ as wanted. Alternatively, just evaluate the integral to see that the whole expression is $\cos(r)$.

(b)

$$\left(r^{-n}J_n(r)\right)' = -r^{-n}J_{n+1}(r),$$

n a non-negative integer.

We do this for both definitions of the Bessel functions. Assuming

$$J_n(r) = \frac{\left(\frac{r}{2}\right)^n}{\Gamma(n+\frac{1}{2})\sqrt{\pi}} \int_{-1}^{1} e^{irs}(1-s^2)^{n-\frac{1}{2}} ds,$$

we calculate

$$\begin{aligned}
(r^{-n}J_n(r))' &= \frac{d}{dr}\left[r^{-n}\frac{\left(\frac{r}{2}\right)^n}{\Gamma(n+\frac{1}{2})\sqrt{\pi}} \int_{-1}^{1} e^{irs}(1-s^2)^{n-\frac{1}{2}} ds\right] \\
&= \frac{d}{dr}\left[\frac{\left(\frac{1}{2}\right)^n}{\Gamma(n+\frac{1}{2})\sqrt{\pi}} \int_{-1}^{1} e^{irs}(1-s^2)^{n-\frac{1}{2}} ds\right] \\
&= \frac{\left(\frac{1}{2}\right)^n}{\Gamma(n+\frac{1}{2})\sqrt{\pi}} \int_{-1}^{1} (is)e^{irs}(1-s^2)^{n-\frac{1}{2}} ds \\
&= \frac{-i\left(\frac{1}{2}\right)^{n+1}}{\Gamma(n+\frac{3}{2})\sqrt{\pi}} \int_{-1}^{1} e^{irs}\left((n+\frac{1}{2})(1-s^2)^{n-\frac{1}{2}}(-2s)\right) ds \\
&= \frac{i\left(\frac{1}{2}\right)^{n+1}}{\Gamma(n+\frac{3}{2})\sqrt{\pi}} \int_{-1}^{1} (ir)e^{irs}(1-s^2)^{n+\frac{1}{2}} ds,
\end{aligned}$$

where in the last step we integrate by parts, and all boundary terms are 0. From this we obtain

$$\begin{aligned}
(r^{-n}J_n(r))' &= \frac{i^2 r\left(\frac{1}{2}\right)^{n+1}}{\Gamma(n+\frac{3}{2})\sqrt{\pi}} \int_{-1}^{1} e^{irs}(1-s^2)^{n+\frac{1}{2}} ds \\
&= -r^{-n}\frac{\left(\frac{r}{2}\right)^{n+1}}{\Gamma(n+\frac{3}{2})\sqrt{\pi}} \int_{-1}^{1} e^{irs}(1-s^2)^{n+\frac{1}{2}} ds \\
&= r^{-n}J_{n+1}(r).
\end{aligned}$$

The second definition of the Bessel functions is

$$J_n(r) = \frac{1}{2\pi}\int_0^{2\pi} e^{ir\sin\theta}e^{-in\theta} d\theta.$$

In this case we calculate

$$
\begin{aligned}
(r^{-n}J_n(r))' &= \frac{d}{dr}\left[\frac{r^{-n}}{2\pi}\int_0^{2\pi} e^{ir\sin\theta}e^{-in\theta}d\theta\right] \\
&= -\frac{r^{-n}}{2\pi}\frac{n}{r}\int_0^{2\pi} e^{ir\sin\theta}e^{-in\theta}d\theta + \frac{r^{-n}}{2\pi}\int_0^{2\pi}(i\sin\theta)e^{ir\sin\theta}e^{-in\theta}d\theta.
\end{aligned}
$$

Meanwhile,

$$
\begin{aligned}
-r^{-n}J_{n+1}(r) &= -\frac{r^{-n}}{2\pi}\int_0^{2\pi} e^{ir\sin\theta}e^{-i(n+1)\theta}d\theta \\
&= -\frac{r^{-n}}{2\pi}\int_0^{2\pi} e^{-i\theta}e^{ir\sin\theta}e^{-in\theta}d\theta \\
&= -\frac{r^{-n}}{2\pi}\int_0^{2\pi}(\cos\theta - i\sin\theta)e^{ir\sin\theta}e^{-in\theta}d\theta \\
&= -\frac{r^{-n}}{2\pi}\int_0^{2\pi}(\cos\theta)e^{ir\sin\theta}e^{-in\theta}d\theta + \frac{r^{-n}}{2\pi}\int_0^{2\pi}(i\sin\theta)e^{ir\sin\theta}e^{-in\theta}d\theta.
\end{aligned}
$$

Attempting to equate this with what we calculated for $(r^{-n}J_n(r))'$, we may cancel the second integral from each expression, as well as a factor of $-\frac{r^{-n}}{2\pi}$. We now use integration by parts on the first term of the second expression:

$$
\begin{aligned}
\int_0^{2\pi}(\cos\theta)e^{ir\sin\theta}e^{-in\theta}d\theta &= \frac{1}{ir}\int_0^{2\pi}\left(ir\cos\theta e^{ir\sin\theta}\right)e^{-in\theta}d\theta \\
&= \left[\frac{1}{ir}e^{ir\sin\theta}e^{-in\theta}\right]_0^{2\pi} + \frac{in}{ir}\int_0^{2\pi} e^{ir\sin\theta}e^{-in\theta}d\theta \\
&= 0 + \frac{n}{r}\int_0^{2\pi} e^{ir\sin\theta}e^{-in\theta}d\theta,
\end{aligned}
$$

where now we use that $n \in \mathbb{N}$ because this forces $e^{-in\cdot 0} = e^{-in\cdot 2\pi} = 1$, making the boundary term evaluate to $\frac{-1}{ir}(1-1) = 0$. Staring at the equations for a moment, we're done.

Remark. For non-negative integers n, there had been two apparently different definitions of $J_n(r)$, and we have seen that both definitions satisfy the same recurrence relation $\frac{d}{dr}(r^{-n}J_n(r)) = -r^{-n}J_{n+1}(r)$. Hence they are both determined by $J_0(r)$. Now it is easy to check that both definitions of $J_0(r)$ agree; just make a change of variable. Hence the two definitions of $J_n(r)$ are actually the same.

6. *Supply the details of the proof of the interpolation theorem.*

Observe that if we have instead $||T_{0+it}(f)||_{q_0} \leq A||f||_{p_0}$ and $||T_{1+it}(f)||_{q_1} \leq B||f||_{p_1}$, the theorem can be applied to $T_s/(A^{1-s}B^s)$ to obtain a sharp bound for T_θ.

Proof. Let simple functions f and g with $||f||_p = ||g||_{q'} = 1$ be given. (Here $'$ denotes the conjugate exponent.) Define

$$f_s = |f|^{\alpha(s)p}\,\mathrm{sgn}(f) \quad \text{and} \quad g_s = |g|^{\alpha(s)q'}\,\mathrm{sgn}(g)$$

where $\alpha(s) = (1-s)/p_0 + s/p_1$, $\beta(s) = (1-s)/q_0' + s/q_1'$, and $\mathrm{sgn}(f)$ is the function such that $f = |f|\,\mathrm{sgn}(f)$. A direct calculation shows that $f_\theta = f$ and that $g_\theta = g$. Let

$$\Psi(s) = \int T_s(f_s) g_s d\mu$$

1)$|\Psi(0+it)| \leq 1$. In fact

$$\begin{aligned} |\Psi(0+it)| \leq & \int |T_{0+it}(|f|^{p/p_0+ip((1-t)/p_0+t/p_1)}\,\mathrm{sgn}(f))|\ |g|^{q'/q_0'}d\mu && \textit{(definition)} \\ \leq & \ |||T_{0+it}(|f|^{p/p_0+ip((1-t)/p_0+t/p_1)}\,\mathrm{sgn}(f))|||_{q_0}|||g|^{q'/q_0'}||_{q_0'} && \textit{(Hölder's} \neq\textit{)} \\ \leq & \ |||f|^{p/p_0}||_{p_0}|||g|^{q'/q_0'}||_{q_0'} = 1 && \textit{(by hypothesis)} \end{aligned}$$

2)$|\Psi(1+it)| \leq 1$. This calculation is the same as the previous one, except that the roles of the subscripts 0 and 1 are interchanged.

3)$\Psi(s)$ is analytic in the strip. In fact, suppose $f = \sum a_n\chi_n$ and $g = \sum b_n\chi_n$, where a_n and b_n are complex coefficients and χ_n are characteristic functions of measurable sets in M. Then we have

$$\Psi(s) = \sum_m \sum_n |a_m|^{\alpha(s)p}\,\mathrm{sgn}(a_m)|b_n|^{\beta(s)q'}\,\mathrm{sgn}(b_n)\int T_s(\chi_m)\chi_n d\mu$$

Here we have used the linearity of T_s to pull the s dependence of the functions f_s and g_s out of the integrand. Since the integral is now analytic in s by hypothesis, we have a sum of analytic functions which is again analytic.

From these three facts it follows that $|\Psi(s)| \leq 1$ in the strip. In particular $|\Psi(\theta)| \leq 1$. We now fix θ and regard $\Psi_g(\theta)$ as a bounded linear functional on $g \in L^{q'}(M)$, where now $|\Psi_g(\theta)| \leq ||g||_{q'}$. This means that the norm of $T_\theta(f)$ as a linear functional on $L^{q'}(M)$ is bounded by 1, from which it follows that $||T_\theta(f)||_q \leq 1$. For f with p-norm different from 1, we apply this last inequality to $f/||f||_p$ and obtain $||T_\theta(f)||_q \leq ||f||_p$ as desired.

□

7. *Let $\mathcal{F}f = \hat{f}$ and suppose*

$$\mathcal{F} : L^p(\mathbb{R}) \to L^q(\mathbb{R}^n) \tag{4}$$

is bounded, say

$$\|\hat{f}\|_q \leq A\|f\|_p \tag{5}$$

where the constant A is independent of f.

(a) *Show that necessarily* $1/p + 1/q = 1$.

Fix $f \in L^p$ such that $\|\hat{f}\|_q \neq 0$. Fix $\delta > 0$. Define $g(x) = f(\delta x)$. We get

$$\|g\|_p = \left(\int |f(\delta x)|^p dx\right)^{1/p} = \left(\int \delta^{-d}|f(x)|^p dx\right)^{1/p} = \delta^{-d/p}\|f\|_p$$

This gives us that $g \in L^p$ and so we know that $\hat{g} \in L^q$. We get

$$\hat{g}(\xi) = \int e^{-2i\pi x\cdot\xi} f(\delta x)dx = \delta^{-d}\int e^{-2i\pi(\delta x)\cdot(\xi/\delta)} f(\delta x)d(\delta x) = \delta^{-d}\hat{f}(\xi/\delta)$$

We get $\|\hat{g}\|_q = \delta^{-d+d/q}\|f\|_q$. Using (5) we get

$$\delta^{-d+d/q}\|\hat{f}\|_q = \|\hat{g}\|_q \leq A\|g\|_p = A\delta^{-d/p}\|f\|_p \tag{6}$$

Using (5) and (6), we get

$$\delta^{d(1/q+1/p-1)} \leq 1 \tag{7}$$

Letting δ in (7) tend towards 0 and ∞, we get that $1/p + 1/q = 1$.

Remark. Such a scaling argument (or dilation argument) is very useful in determining the only possible relations between the exponents of an inequality that holds for 'all' functions. For instance, except with a twist, this argument shows that in the restriction theorem $\|\hat{f}\|_{L^q(\mathbb{S}^{n-1})} \leq C\|f\|_{L^p(\mathbb{R}^n)}$ for the sphere, the only possible exponents are those (p, q) which satisfy $q = \frac{(n-1)p'}{n+1}$.

(b) *Show that necessarily* $1 \leq p \leq 2$.

Consider $f(x) = e^{-\pi\delta|x|^2}$ where $\delta = \alpha + i\beta$, $\alpha > 0, \beta \in \mathbb{R}$. Then

$$\|f\|_p = \left(\int |e^{-\pi\delta|x|^2}|^p dx\right)^{1/p} = \left(\int e^{-\pi\alpha p|x|^2} dx\right)^{1/p} = (\alpha p)^{-\frac{d}{2p}} \tag{8}$$

Also

$$\begin{aligned}
\|\hat{f}\|_q &= \left(\int \left||\delta|^{-d/2}e^{-\pi|x|^2/\delta}\right|^q dx\right)^{1/q} \\
&= \left(\int \left||\delta|^{-d/2}e^{-\pi|x|^2\alpha/|\delta|^2}\right|^q dx\right)^{1/q} \\
&= \left(\int |\delta|^{-dq/2}e^{-\pi|x|^2 q\alpha/|\delta|^2} dx\right)^{1/q} \\
&= |\delta|^{-d/2}\left(\alpha q|\delta|^{-2}\right)^{-d/(2q)}\left(\int e^{-\pi|x|^2}dx\right)^{1/q}
\end{aligned} \tag{9}$$

Thus

$$\begin{aligned}
|\delta|^{-d/2+d/q}\alpha^{-d/(2q)}q^{-d/(2q)} &\leq A\alpha^{-d/(2p)}p^{-d/(2p)} \\
A^{-1}\alpha^{d/(2p)-d/(2q)}q^{-d/(2q)}p^{d/(2p)} &\leq |\delta|^{d(1/2-1/q)}
\end{aligned} \tag{10}$$

by fixing $\alpha > 0$ and letting $\beta \to \infty$, we get $|\delta| \to \infty$. To get (10) to hold, we must have $q \geq 2$, which is just $1 \leq p \leq 2$ (by part a).

8. *Carry out the proof (via polar coordinates on the sphere $\mathbb{S}^{d-1}$) that:*

(a)

$$\widehat{d\sigma}(\xi) = 2\frac{\sin(2\pi|\xi|)}{|\xi|}, \quad d = 3.$$

Observe that since $d\sigma$ is radial, $\widehat{d\sigma}$ is also radial. Thus $\widehat{d\sigma}(\xi) = \widehat{d\sigma}(0,0,|\xi|)$. Write for $x \in \mathbb{S}^2$ that $x = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$. Then

$$\begin{aligned}
\widehat{d\sigma}(0,0,|\xi|) &= \int_{\mathbb{S}^2} e^{-2\pi i x_3|\xi|} d\sigma(x) \\
&= \int_0^\pi \int_0^{2\pi} e^{-2\pi i|\xi|\cos\phi} \sin\phi d\theta d\phi \\
&= 2\pi \left. \frac{e^{-2\pi i|\xi|\cos\phi}}{2\pi i|\xi|} \right|_0^\pi \\
&= 2\frac{\sin(2\pi|\xi|)}{|\xi|}.
\end{aligned}$$

(b) *More generally,*

$$\widehat{(d\sigma)}(\xi) = 2\pi|\xi|^{-\frac{d-2}{2}} J_{\frac{d-2}{2}}(2\pi|\xi|)$$

using the formula for the Bessel function J_k, for $k > -1/2$.

Again, $d\sigma$ is radial. Therefore

$$\begin{aligned}
\widehat{d\sigma}(\xi) &= \int_{\mathbb{S}^{d-1}} e^{-2\pi i x_d|\xi|} d\sigma(x) \\
&= \int_0^\pi \int_{\mathbb{S}^{d-2}} e^{-2\pi i|\xi|\cos\phi} (\sin\phi)^{d-2} d\sigma_{d-2} d\phi \\
&= |\mathbb{S}^{d-2}| \int_{-1}^1 e^{-2\pi i|\xi|s} (1-s^2)^{\frac{d-2}{2}} ds \\
&= 2\pi|\xi|^{-\frac{d-2}{2}} J_{\frac{d-2}{2}}(2\pi|\xi|).
\end{aligned}$$

where the second to last equality makes use of the change of variable $s = \cos\phi$.

9. Give an example of a C^∞ closed curve in $\mathbb{R}^2$ to that if $d\sigma$ is the arc-length measure on it, then $\widehat{d\sigma}(\xi) \neq o(1)$ as $|\xi| \to \infty$.

Indeed any C^∞ closed curve that contains a straight line segment cannot have $\widehat{d\sigma} = o(1)$ as $|\xi| \to \infty$, if $d\sigma$ is the arc-length measure on it. This is because we can consider a measure $d\mu$ on the curve given by

$$d\mu(x) = f(x) d\sigma(x),$$

where f is a compactly supported smooth function on $\mathbb{R}^2$ whose support only intersects the curve in the straight line segment. Then

$$\widehat{d\mu}(\xi) = \widehat{f} * \widehat{d\sigma}(\xi).$$

Assume on the contrary that $\widehat{d\sigma}(\xi) = o(1)$ as $|\xi| \to \infty$. Then since $\hat{f}(\xi) = o(1)$ as $|\xi| \to \infty$ as well, we have $\widehat{d\mu}(\xi) = o(1)$ as $|\xi| \to \infty$. But this is impossible, since $d\mu$, being a finite measure supported on a line segment, has Fourier transform being constant along straight lines that are perpendicular to that segment. This proves our claim.

10. *Consider $J_n(r)$ with n integral. Show*

   (a) $|J_n(r)| \leq Ar^{-1/2}$*, uniformly in n and r, if $r > cn$, for any fixed $c > 1$.*
   We will use the following van der Corput estimate (see p334 of *Harmonic Analysis* by E. M. Stein): If ϕ is real valued and C^k in (a, b), and satisfies $|\phi^{(k)}| \geq 1$ (and ϕ' monotonic for $k = 1$), then we have

$$\Big| \int_a^b e^{i\lambda\phi(x)} dx \Big| \leq C_k \lambda^{-1/k}$$

   C_k independent of a, b and ϕ. Because n is an integer, we use the formula

$$J_n(r) = \frac{1}{2\pi} \int_0^{2\pi} e^{ir\sin\theta} e^{-in\theta} d\theta.$$

   Observe that if in the hypotheses of the estimate we have $|\phi^{(k)}| \geq \varepsilon$ rather than $|\phi^{(k)}| \geq 1$, this only changes the leading constant by a factor of $\varepsilon^{-1/k}$. Now r is in the role of λ, and $f(\theta) = \sin\theta - \frac{n}{r}\theta$ is in the role of ϕ. Using $r > cn$, we see that $f'(\theta) > \cos\theta - 1/c > \varepsilon_c$ if $\theta \in [0, \theta_c] \cup [2\pi - \theta_c, 2\pi]$, where θ_c is a constant depending on c only. Note also $f'(\theta) = \cos\theta - n/r < -1/2$ if $|\theta - \pi| < \pi/3$. Hence on these intervals, the total contribution of the integrals are $16(\varepsilon_c r)^{-1}$, a decay better than desired. (Caution: We actually have to divide the integral from $2\pi/3$ to $4\pi/3$ into two halves, namely from $2\pi/3$ to π and then from π to $4\pi/3$, because the phase function is not monotonic on the big interval. That does not harm our estimate though.) Now $f''(\theta) = \sin\theta$, so over the rest of $[0, 2\pi]$ that is not covered above, we have $|f''(\theta)| > \varepsilon_c$. Apply van der Corput's estimate on each of the intervals, we get the bound $C_2(\varepsilon_c r)^{-1/2}$, so overall we get a bound $A_c r^{-1/2}$ for $J_n(r)$, with A_c depending only on c but not on n nor r.

   (b) $|J_n(r)| \leq An^{-N}$*, for each N, if $n > cr$, for any fixed $c > 1$.*
   For this we use the nonstationary phase principle. Let $f(\theta) = \theta - \frac{r}{n}\sin\theta$, and observe that

$$f'(\theta) = 1 - \frac{r}{n}\cos\theta \geq 1 - \frac{r}{n} \geq 1 - \frac{1}{c}.$$

   This allows us to apply repeated integrations by parts in the following manner:

$$\int_0^{2\pi} e^{inf(\theta)} d\theta = \int_0^{2\pi} e^{inf(\theta)} \frac{inf'(\theta)}{inf'(\theta)} d\theta = e^{inf(\theta)} \frac{1}{inf'(\theta)}\Big|_{\theta=0}^{2\pi} - \int_0^{2\pi} e^{inf(\theta)} \left(\frac{1}{inf'(\theta)}\right)' d\theta.$$

   The boundary terms cancel because $f(2\pi) - f(0) = 2\pi$, while $f'(\theta)$ is 2π-periodic. The new integrand is now manifestly $O(n^{-1})$. To obtain better powers of n, we repeat this integration by parts. The boundary terms are of the form $e^{inf(\theta)}g(\theta)$, where $g(\theta)$ is a function of $f'(\theta)$ and its derivatives, and hence continue to give no contribution. Meanwhile each successive integration by parts gives an additional factor of n^{-1} in the integral.

(c) $|J_n(r)| \leq Ar^{-1/3}$, *uniformly for all* n *and* r.

Here we again use r for λ, and $f(\theta) = \sin\theta - \frac{n}{r}\theta$ for ϕ. This time we observe that $|f''(\theta)| + |f'''(\theta)| \geq c$ for some $c > 0$ independent of n and r. We can now decompose $(0, 2\pi)$ into subintervals, on each of which we have a lower bound on either $|f''|$ or on $|f'''|$. We break up our integral into several pieces using a partition of unity as before, and this time each piece is bounded by either $Cr^{-1/2}$ or by $Cr^{-1/3}$. So overall we get $Ar^{-1/3}$, A independent of n and r.

11. *Let* $I_0(\lambda) = \displaystyle\int_a^b e^{i\lambda\Phi(x)}dx$. *In obtaining the estimate* $I_0(\lambda) = O(\lambda^{-1})$, *show that the condition that* Φ' *is monotonic cannot be relaxed.*

Let $\lambda = 1$. The real part of the integrand is then $\cos(\Phi(x))$. Suppose that $\Phi' \geq 1$ oscillates so that it is large when $\cos(\Phi(x)) < 0$ and is small when $\cos(\Phi(x)) > 0$. This implies that $\Phi(x)$ escapes quickly when $\cos(\Phi(x)) < 0$ and changes slowly when $\cos(\Phi(x)) > 0$. Hence the measure of the set where $\cos(\Phi(x)) > 0$ is much larger than the measure of the set where $\cos(\Phi(x)) < 0$. Thus the real part of the integral is unbounded as $b \to \infty$.

12. *Write out a proof of "Morse's Lemma": If* $\Phi(0) = |\nabla\Phi(0)| = 0$ *and* $\nabla^2\Phi(0)$ *has nonvanishing determinant, then there is a smooth change of variables* $x \to y$, *so that near the origin* $\Phi(x) = y_1^2 + y_2^2 + \cdots + y_q^2 - (y_{q+1}^2 + \cdots + y_d^2)$.

Lemma (Morse's Lemma)**.** *Given that 0 is a nondegenerate critical point of a smooth real function* f *on a manifold* M, *that is* $|\nabla f(0)| = 0$ *and* $\det(\nabla^2 f(0)) \neq 0$, *and that* $f(0) = 0$, *there is a local coordinate system* (y_i) *such that* $f(y) = y_1^2 + y_2^2 + \cdots + y_q^2 - (y_{q+1}^2 + \cdots + y_d^2)$ *around the origin.*

In the proof we use the inverse function theorem and the following simple lemma.

Lemma (Hadamard's Lemma)**.** *Let* $f : U \to \mathbb{R}$ *be* C^k *for some* $k \geq 1$ *defined on a convex neighborhood* U *of* $0 \in \mathbb{R}^m$, *and* $f(0) = 0$. *The there exist functions* $g_i \in C^{k-1}$, $i = 1, \ldots, m$ *defined on* U *such that* $f(x_1, \ldots, x_m) = \sum_{i=1}^m x_i g_i(x_1, \ldots, x_m)$ *and* $g_i(0) = \frac{\partial f}{\partial x_i}(0)$.

Proof. Note that

$$f(x_1, \ldots, x_m) = \int_0^1 \frac{df(tx_1, \ldots, tx_m)}{dt}dt = \int_0^1 \sum_{i=1}^m \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_m)x_i dt$$

Thus $g_i(x_1, \ldots, x_m) = \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_m)dt$ satisfies the required conditions. □

Proof of Morse's Lemma. Let $(x_1, \ldots, x_m)$ be a coordinate neighborhood around 0. By the above lemma, we can write $f(x_1, \ldots, x_m) = \sum_{i=1}^m x_i g_i(x_1, \ldots, x_m)$. Since we have $|\nabla f| = 0$, $g_i(0) = \frac{\partial f}{\partial x_i}(0) = 0$. Thus we can apply Hadamard's lemma to each g_i, getting $g_i(x_1, \ldots, x_m) = \sum_{j=1}^m x_j h_{ij}(x_1, \ldots, x_m)$ and

$$f(x_1, \ldots, x_m) = \sum_{i,j=1}^m x_i x_j h_{ij}(x_1, \ldots, x_m)$$

Without loss of generality, assume $h_{ij} = h_{ji}$. Otherwise we could define $\tilde{h}_{ij} = \frac{1}{2}(h_{ij} + h_{ji})$, resulting in $\tilde{h}_{ij} = \tilde{h}_{ji}$ and $f = \sum x_i x_j \tilde{h}_{ij}$. Note that by Hadamard's lemma, we have $h_{ij}(0) = \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}(0)$, so $(h_{ij}(0))$ is nonsingular.
We now proceed by induction. Suppose there is a neighborhood $U_1 \subset U$ parametrized by coordinates (u_i) around 0 and a diffeomorphism ϕ, $x_i = \phi^i(u_1, \dots, u_m)$ such that

$$(f \circ \phi)(u) = \pm u_1^2 \pm u_2^2 \pm \cdots \pm u_{r-1}^2 + \sum_{i,j \geq r} u_i u_j H_{ij}(u_1, \dots, u_m) \tag{11}$$

and (H_{ij}) symmetric for all $u \in U_1$. Note that since $(h_{ij}(0))$ is nonsingular and ϕ is a diffeomorphism, we have

$$0 \neq \det \left(\phi'(0)^T (h_{ij}(0)) \phi'(0)\right)$$

where the right hand side is equal to the matrix form of (11) with H_{ij} evaluated at 0. Thus at least one entry of $H_{ij}(0)$, $i, j \geq r$ is nonzero and by a linear change of the last $n - r$ coordinates we can make $H_{rr}(0) \neq 0$. By continuity, $H_{rr}(u) \neq 0$ in some neighborhood $U_2 \subset U_1$ of 0. Let $g(u_1, \dots, u_m) = \sqrt{|H_{rr}(u)|}$, a function definted on U_2. Consider the following coordinate change:

$$\begin{aligned} v_i &= u_i & i &\neq r \\ v_r &= g(u_1, \dots, u_m)\left(u_r + \textstyle\sum_{i>r} \frac{u_i H_{ir}(u_1, \dots, u_m)}{H_{rr}(u_1, \dots, u_m)}\right) & i &= r \end{aligned}$$

The Jacobian of this transformation at $u = 0$ is simply $g(0) \neq 0$ so by the inverse function theorem there is a neighborhood $U_3 \subset U_2$ of 0 on which the above coordinate change, which we denote $v = \psi(u)$, is a diffeomorphism. Note that now we have

$$v_r v_r = \pm H_{rr}(u) u_r u_r \pm 2 \sum_{i>r} u_r u_i H_{ri}(u) + \sum_{i,j>r} \frac{u_i u_j H_{ir}(u) H_{jr}(u)}{|H_{rr}(u)|}$$

so that the u_r terms in (11) can be replaced by $v_r v_r$ minus a sum over indices larger than r, leading to

$$(f \circ \phi \circ \psi^{-1})(v) = \pm v_1^2 \pm \cdots \pm v_r^2 + \sum_{i,j>r} v_i v_j \tilde{H}_{ij}(v_1, \dots, v_m)$$

with $\tilde{H}_{ij}$ smooth and symmetric. This completes the induction step of the proof.

□

Remark. If Φ is a homogeneous quadratic polynomial and $\nabla^2\Phi(0)$ non-degenerate, then one can diagonalize Φ rather easily. The point is even for a general Φ, as long as $\Phi(0) = |\nabla\Phi(0)| = 0$, we can write it as in the Hadamard's lemma such that it looks like a 'variable coefficient quadratic polynomial'. The proof for the case of a homogeneous quadratic polynomial then carry through, as long as we are working near 0, where the coefficients can be thought of as 'roughly constant'.

13. *Show that the averages theorem (in $\mathbb{R}^3$, $A : L^{4/3} \to L^4$) cannot be improved.*

Let $\varepsilon > 0$ be given. We will find $f \in L^{4/3}(\mathbb{R})$ such that $A(f) = \int_{S^2} f(x - y) d\sigma(y) \notin L^{4+\varepsilon}(\mathbb{R})$.

We first observe that $\chi_{B(0,2)}|x|^{-\alpha} \in L^1(\mathbb{R}^3) \Leftrightarrow 3 > \alpha$. So

$$\chi_{B(0,2)}|x|^{-\alpha} \in L^{4/3}(\mathbb{R}^3) \text{ iff } \alpha < 9/4 \tag{12}$$

We will take $f(x) = \chi_{B(0,2)}|x|^{-\alpha}$, where $\alpha < 9/4$ is to be determined later. Then $f \in L^{4/3}$ by (12). Now we observe that $Af(x)$ is radial. In fact, if $\rho : \mathbb{R}^3 \to \mathbb{R}^3$ is a rotation, then

$$\begin{aligned} Af(\rho x) &= \int_{S^2} \frac{\chi_{B(0,2)}(\rho x - y)}{|\rho x - y|^\alpha} d\sigma(y) = \int_{S^2} \frac{\chi_{B(0,2)}(\rho x - \rho z)}{|\rho x - \rho z|^\alpha} d\sigma(z) \\ &= \int_{S^2} \frac{\chi_{B(0,2)}(x - z)}{|x - z|^\alpha} d\sigma(z) = Af(x) \end{aligned}$$

Let $r \in (0,3)$ be given. Let $x = (0,0,r)$. Then, using spherical coordinates, we get, for $\alpha \neq 2$,

$$\begin{aligned} Af(x) &= \int_{S^2} \frac{d\sigma(y)}{|x-y|^\alpha} = \int_0^{2\pi}\int_0^\pi \frac{\sin\varphi d\varphi d\theta}{(1-2r\cos\varphi + r^2)^{\alpha/2}} = 2\pi \int_0^\pi \frac{\sin\varphi d\varphi}{(1-2r\cos\varphi+r^2)^{\alpha/2}} \\ &= \frac{2\pi}{2r}\int_{(1-r)^2}^{(1+r)^2} \frac{du}{u^{\alpha/2}} = \frac{\pi((1+r)^{2-\alpha} - (1-r)^{2-\alpha})}{(1-\alpha/2)r} \end{aligned}$$

We expect Af to be very large when $|x| \approx 1$. Now the function $x \mapsto (\frac{1}{|x|}(1+|x|)^{2-\alpha})^{4+\varepsilon}$ is integrable for $|x|$ near 1. So, we want to find α such that $\int_{|x|\in[1/2,1]}(\frac{1}{|x|}(1-|x|)^{2-\alpha})^{4+\varepsilon}dx = +\infty$. In spherical coordinates, this integral is

$$C\int_{1/2}^1 (\frac{1}{r}(1-r))^{(2-\alpha)(4+\varepsilon)} r^2 dr \geq D\int_{1/2}^1 (1-r)^{(2-\alpha)(4+\varepsilon)} r^2 dr$$

Now

$$\int_{1/2}^1 (1-r)^{(2-\alpha)(4+\varepsilon)} r^2 dr \geq \frac{1}{4}\int_{1/2}^1 (1-r)^{(2-\alpha)(4+\varepsilon)} dr = \frac{1}{4}\int_0^{1/2} s^{(2-\alpha)(4+\varepsilon)} ds$$

If $(2-\alpha)(4+\varepsilon) = -1$, then this integral is $+\infty$ and $Af \notin L^{4+\varepsilon}(\mathbb{R}^3)$. However, then $\alpha = 2 + \frac{1}{4+\varepsilon}$, so $f \in L^{4/3}(\mathbb{R}^3)$.

Alternative solution. Let $f(x) = \chi_{B(0,\varepsilon)}(x)$ in $\mathbb{R}^d$. Then $\|f\|_{L^{(d+1)/d}} = c\varepsilon^{\frac{d^2}{d+1}}$. Note $Af(x) \simeq c\varepsilon^{d-1}$ when $||x|-1| < \varepsilon/2$. Hence $\|Af\|_{L^p} \geq c\varepsilon^{d-1}\left|\{||x|-1| < \varepsilon/2\}\right|^{\frac{1}{p}} = c\varepsilon^{d-1+\frac{1}{p}}$ and for $c\varepsilon^{d-1+\frac{1}{p}} \leq C\varepsilon^{\frac{d^2}{d+1}}$ to hold as $\varepsilon \to 0$ we need $d-1+\frac{1}{p} \geq \frac{d^2}{d+1}$, i.e. $p \leq d+1$. So A cannot map $L^{\frac{d+1}{d}}$ to any L^p where $p > d+1$. (The idea here is that characteristic functions are relatively easy to average, and that simplifies the calculation.)

14. *Show that the spherical maximal theorem fails for* $p \leq \frac{d}{d-1}$.

We will construct counterexamples depending on p and d, and in each case the spherical maximal function of our counterexample will be everywhere infinite. If $d = 1$, the spherical maximal theorem fails for all $p < \infty$. Indeed, consider $f(x) = \chi(x)|x|^{-\varepsilon} \in L^p(\mathbb{R})$, where $\varepsilon > 0$ is sufficiently small and χ is any compactly supported function which is positive near the origin. The spheres in this case are pairs of points, and as one of the points approaches the origin we see that the absolute value of the average increases without bound.

We now assume $d \geq 2$, and treat first the case $p < \frac{d}{d-1}$, so fix $p \in [1, \frac{d}{d-1})$, and put

$$f(x) = \chi(x)|x|^{-\frac{d}{p}} \left|\log^{-\frac{2}{p}} |x|\right|,$$

where χ is the characteristic function of the ball centered at 0 with radius 1/2. We first verify that this function is in L^p by computing as follows:

$$\int |f|^p = \int \chi(x)|x|^{-d} \log^{-2} |x| = c \int_0^{1/2} \frac{dr}{r \log^2 r} = c \int_{-\infty}^{\log 1/2} s^{-2} ds < \infty,$$

where we used polar coordinates, followed by the change of variables $\log r = s$.

We now estimate from below the spherical maximal function at a point x_0. Fix $x_0 \in \mathbb{R}^d \backslash \{0\}$, $\varepsilon \in (0, \min\{1/2, |x_0|\})$, and let D_ε denote the intersection of the sphere centered at x_0 with radius $|x_0| - \frac{\varepsilon}{2}$ with the ball centered at 0 with radius ε. We observe that the area of D_ε is bounded below by $c\varepsilon^{d-1}$, where c is a constant proportional to the area of the unit ball in $\mathbb{R}^{d-1}$. On the other hand $\frac{\varepsilon}{2} \leq |x| \leq \varepsilon$ for $x \in D_\varepsilon$, so we may write

$$\int_S f d\sigma \geq \int_{D_\varepsilon} f d\sigma \geq c\varepsilon^{d-1} \varepsilon^{-\frac{d}{p}} \left|\log^{-\frac{2}{p}} \frac{\varepsilon}{2}\right|.$$

But $d-1-\frac{d}{p} < 0$ for $p \in [1, \frac{d}{d-1})$, so letting $\varepsilon \to 0$ shows that the spherical maximal function is infinite at x_0.

If $p = \frac{d}{d-1}$, a more delicate analysis is necessary. We put

$$f(x) = \chi(x)|x|^{-d+1} \left|\log^{-1} |x|\right|,$$

and observe that, by the same reasoning as before, $f \in L^p$. To see why the spherical maximal function is infinite, consider the following heuristic argument: Let $D_{\delta,\varepsilon}$ denote the intersection of the sphere centered at x_0 with radius $|x_0| - \delta$ with the ball centered at 0 with radius ε. For a fixed $\varepsilon > 0$, $\varepsilon << |x_0|$, as $\delta \to 0^+$ the integral over $D_{\delta,\varepsilon}$ resembles more and more closely the integral over a disk of codimension 1 centered at zero and with radius ε. We then have

$$\int_S f d\sigma \geq \int_{D_\varepsilon} f d\sigma \sim c \int_0^\varepsilon f(r) r^{d-2} dr = c \int_0^\varepsilon \frac{dr}{r \log r} = \infty.$$

The technical difficulty lies in justifying the '$\sim$'. We give a brute force treatment of the integral here without quite following the above line of reasoning, but if anyone knows a simpler proof, I'd love to hear it. For now, consider the sphere centered at x_0 with radius $|x_0|$, and parametrize the portion of this sphere near the origin using $\alpha \overset{\text{def}}{=} \measuredangle(0, x_0, x)$. Observe that the surface measure on the sphere is given in terms of α by $\int f(\alpha) d\sigma = c|x_0|^{d-1} \int f(\alpha)(\sin\alpha)^{d-2} d\alpha$, which we write more simply as $\int f(\alpha) d\sigma = c \int f(\alpha)(\sin\alpha)^{d-2} d\alpha$

(the constants here will be allowed to depend on x_0). Observe that, by the law of sines, we have $|x| = |x_0| \frac{\sin\alpha}{\cos\frac{\alpha}{2}} = c\sin\frac{\alpha}{2}$. Putting this together we see that

$$\begin{aligned}\int_S f d\sigma &\geq c\int_0^\varepsilon f\left(c\sin\frac{\alpha}{2}\right)(\sin\alpha)^{d-2}d\alpha \\ &= c\int_0^\varepsilon \frac{(\sin\alpha)^{d-2}}{\left(\sin\frac{\alpha}{2}\right)^{d-1}\log\left(c\sin\frac{\alpha}{2}\right)}d\alpha \\ &= c\int_0^\varepsilon \frac{(\cos\frac{\alpha}{2})^{d-2}}{\sin\frac{\alpha}{2}\log\left(c\sin\frac{\alpha}{2}\right)}d\alpha.\end{aligned}$$

Here we have restricted to a neighborhood of the origin, and simplified our integral using double angle formulas. We now use the substitution $u = c\sin\frac{\alpha}{2}$:

$$= c\int_0^{c\sin\frac{\varepsilon}{2}} \frac{(1-u^2)^{\frac{d-3}{2}}}{u\log u}du.$$

This last integral is divergent, which proves that the spherical maximal function is infinite.

15. *Suppose that S is a smooth hypersurface in $\mathbb{R}^d$ given as a graph $S = \{x : x_d = F(x'),$ $x' \in \mathbb{R}^{d-1}\}$ with F smooth. Verify the formula giving the induced Lebesgue measure: that for any continuous f of compact support*

$$\lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_{S_\epsilon} f(x)dx = \int_{\mathbb{R}^{d-1}} f(x', F(x'))(1+|\nabla_{x'}F|^2)^{1/2}dx'$$

where $S_\epsilon = \{x : d(x,S) < \epsilon\}$

We first show that

$$S_\epsilon = \{(x', F(x')) + t\frac{(\nabla_{x'}F(x'), -1)}{(1+|\nabla_{x'}F(x')|^2)^{1/2}} : x' \in \mathbb{R}^{d-1}, t \in (-\epsilon, \epsilon)\}$$

This means that S_ϵ is a tubular neighborhood of S obtained by taking points $(x', F(x'))$ of S and adding multiples of $\Phi(x') = \frac{(\nabla_{x'}F(x'),-1)}{(1+|\nabla_{x'}F(x')|^2)^{1/2}}$, the unit normal vector to S at $(x', F(x'))$. We first observe that S_ϵ contains this set because each point $(x', F(x')) + t\Phi(x')$ in it is no more than ϵ away from the point $(x', F(x'))$. To see the reverse containment, let a point in S_ϵ be given, and suppose we have chosen our coordinates so that this point is the origin. We will show that it is of the form $(x', F(x')) \pm \delta\Phi(x')$, where $x_0 = (x', F(x'))$ is any point at which the minimal distance is attained and δ is such that $|x_0| = d(0, S) = \delta$. This is the same as showing that the vector pointing to x_0 is perpendicular to S, i.e. that it is perpendicular to any curve in S passing through x_0. To see this, let $x(t)$ be a parametrization of a curve passing through x_0 at $t = 0$, and observe that because $t = 0$ is a local minimum of $|x(t)|^2$ we have $0 = \frac{d}{dt}\big|_{t=0}|x(t)|^2 = 2x'(0)\cdot x(0)$. And this dot product being zero expresses exactly the fact that the vector pointing to x_0 is perpendicular to $x(t)$.

Now let $g(x', t) = (x', F(x')) + t(\Phi_1(x'), \Phi_2(x'))$ and put $\phi(x') = (1+|\nabla_{x'}F(x')|^2)^{1/2}$. The Jacobian of g is given by

$$J(g) = \left(\begin{array}{c|c} \mathrm{Id} + tJ(\Phi_1) & \nabla F + t\nabla\Phi_2 \\ \hline \nabla F/\phi(x') & -1/\phi(x') \end{array}\right)$$

where $J(\Phi)$ is the Jacobian of Φ. Now we rewrite our integral using the change of coordinates given by g

$$\int_{S_\epsilon} f(x)dx = \int_{g(\mathbb{R}^{d-1}\times(-\epsilon,\epsilon))} f(x)dx = \int_{\mathbb{R}^{d-1}}\int_{-\epsilon}^{\epsilon} f((x', F(x')) + t\Phi(x'))|\det(J(g))|dtdx'$$

We now divide this by 2ϵ and let $\epsilon \to 0$. But the fact that f and $J(g)$ are continuous and f is compactly supported allows us to pass the limit through the integral in x', so that we get

$$\lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} f((x', F(x')) + t\Phi(x'))|\det(J(g))|dt = f(x', F(x'))|\det(J(g))|\Big|_{t=0}$$

But an argument by induction on the dimension d shows us that

$$|\det(J(g))|\Big|_{t=0} = \frac{1 + |\nabla F(x')|^2}{\phi(x')} = (1 + |\nabla F(x')|^2)^{1/2}$$

which gives

$$\lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_{S_\epsilon} f(x)dx = \int_{\mathbb{R}^{d-1}} f(x', F(x'))(1 + |\nabla F(x')|^2)^{1/2}dx'$$

as desired.

16. *Verify that the intrinsic definition of Gauss curvature (in terms of the Gauss map of the normals to the unit sphere) agrees with the coordinate dependent definition given for graphs.*

We will do this by computing the Gauss curvature according to the intrinsic definition, and then verifying that this agrees with the definition for graphs.

Let $\mathbf{n}(x) : S \to \mathbb{R}^d$ map each point in the given surface S to a unit normal vector at $\mathbf{n}$; locally this is well defined up to sign. Recall that the directional derivative of the normal vector at a point $\mathbf{n}(x_0)$ in the direction of the tangent vector $\mathbf{v}$ is given by $\nabla\mathbf{n}(x_0)\cdot\mathbf{v}$ where the matrix $\nabla\mathbf{n}(x_0)$ has entry (i, j) given by $\partial_j n_i(x_0)$. Observe that this matrix maps tangent vectors to tangent vectors. In fact, if $\gamma(t)$ is a curve in S passing through x_0 at time zero, then we have

$$2\mathbf{n}(\gamma(0))\cdot\nabla\mathbf{n}(\gamma(0))\cdot\gamma'(0) = \frac{d}{dt}\Big|_{t=0}|\mathbf{n}(\gamma(t))|^2 = 0$$

The first equality follows from the chain rule, and the second from the fact that all the normal vectors have unit length. Intrinsically, the Gauss curvature is defined to be the determinant of this matrix as a map from the tangent space to itself. In odd dimensions this is only uniquely defined up to sign, because our normal vector was only uniquely defined up to sign, and because $\det(-A) = (-1)^n\det A$.

We now compute this explicitly in coordinates. Suppose first we wish to compute the Gauss curvature at the origin and that the surface is given by $S = \{x_d = f(x_1, \dots, x_{d-1})\}$ here with $f(0) = 0$ and $\nabla f(0) = 0$. We will reduce the general case to this one later. Let $\rho : \mathbb{R}^d \to \mathbb{R}^d$ be given by $\rho(x) = x_d - f(x_1, \dots, x_{d-1})$. Then $S = \rho^{-1}(0)$, and a unit normal vector may be defined by $\mathbf{n}(x) = \frac{\nabla\rho(x)}{|\nabla\rho(x)|}$. But $|\nabla\rho(x)| = \sqrt{1 + |\nabla f(x_1, \ldots, x_{d-1})|^2}$, so $|\nabla\rho(0)| = 1$ and $\mathbf{n}(0) = (0, \ldots, 0, 1)$.

Now

$$(\nabla\mathbf{n}(x))_{i,j} = \partial_j n_i(x) = \partial_j \frac{\partial_i \rho(x)}{|\nabla\rho(x)|}$$

We are interested in this matrix as a map from the tangent space at the origin to the tangent space at the origin, so we only allow i and j to range from 1 to $d-1$. This gives

$$(\nabla\mathbf{n}(0))_{i,j} = -\frac{\partial_j\partial_i f(0)}{|\nabla\rho(0)|} + \frac{\partial_i f(0)\partial_j \left.|\nabla\rho(x)|\right|_{x=0}}{|\nabla\rho(0)|^2} = -\partial_j\partial_i f(0)$$

The first term simplifies because $|\nabla\rho(0)| = 1$, and the second vanishes because $\nabla f(0) = 0$.

The determinant of this matrix is thus exactly the determinant of the Hessian of f, up to a sign in odd dimensions, which is exactly coordinate dependent definition of Gauss curvature. To reduce to the general case to the case just solved, observe that we can find a translation τ and a transformation $M \in SL(\mathbb{R}^{d-1})$ such that τ sends x_0 to 0 and M reorients the surface so that it is the graph of a function f with $f(0) = 0$ and $\nabla f(0) = 0$. We must check that the determinant of the Hessian of f at x_0 equals the determinant of the Hessian of $f \circ \tau^{-1} \circ M^{-1}$ at 0.

$$\begin{aligned}\partial_i\partial_j(f \circ \tau^{-1} \circ M^{-1})(0) &= \partial_i \left.\left(\sum_k (\partial_k f) \circ \tau^{-1} \circ M^{-1}(x) \cdot M^{-1}_{k,j}(x)\right)\right|_{x=x_0} \\ &= \sum_{k,\ell} \partial_\ell\partial_k f(x_0) M^{-1}_{\ell,i} M^{-1}_{k,j}\end{aligned}$$

This means, if ∇^2 denotes the Hessian, that $\nabla^2(f \circ \tau^{-1} \circ M^{-1})(0) = M^{-1}\nabla^2 f(x_0) M^{-1}$. Taking the determinant of both sides and using the fact $\det M^{-1} = 1$ we find that the two determinants match.

17. *Consider $I(s) = \int_0^\infty u^{s-1} f(u)du$, where $f \in C^\infty(\mathbb{R})$ and of compact support. Show that the residue of the meromorphic continuation of $I(s)$ at $s = -k$ is $\frac{f^{(k)}(0)}{k!}$*

We want to calculate the residue of $I(s), s \in \mathbb{C}$ for each $-k$, $k = 0, 1, 2, \ldots$, where $I(s)$ is given by (13)

$$I(s) = \int_0^\infty u^{s-1} f(u)du. \tag{13}$$

Here $f \in C_c^\infty$ and $\Re s > 0$. Our first goal is to ensure that we have a meromorphic continuation of the function over the proper domain and then to calculate the residue. If we consider $sI(s), \Re s > 0$, we get (14)

$$sI(s) = s\int_0^\infty u^{s-1} f(u)du = u^s f(u)|_0^\infty - \int_0^\infty u^s f^{(1)}(u)du = -\int_0^\infty u^s f^{(1)}(u)du \tag{14}$$

So, we now have a definition of $I(s), 0 \geq \Re s > -1$ given by (15)

$$I(s) = \frac{-1}{s} \int_0^\infty u^s f^{(1)}(u) du. \tag{15}$$

Continuing by a similar argument, we get the following definition for $I(s), -k \geq \Re s > -(k+1), k = 0, 1, 2, \ldots$ given by (16)

$$I(s) = \left[\prod_{j=0}^{k} \left(\frac{-1}{s+j} \right) \right] \int_0^\infty u^{s+k} f^{(k+1)}(u) du \tag{16}$$

We notice two simple facts about the definition 16. First, the function $I(s)$ is the meromorphic extension of our original $I(s)$. Second, $I(s)$ has simple poles at $-k, k = 0, 1, 2, \ldots$. So, in order to calculate the residue of $I(s)$ at $s = -k$ we must simply compute (17)

$$\lim_{s \to -k} (s+k) I(s). \tag{17}$$

And here is the computation:

$$\begin{aligned}
&\lim_{s \to -k} (s+k) I(s) \\
=& \lim_{s \to -k} (s+k) \left[\prod_{j=0}^{k} \left(\frac{-1}{s+j} \right) \right] \int_0^\infty u^{s+k} f^{(k+1)}(u) du \\
=& \lim_{s \to -k} (-1) \left[\prod_{j=0}^{k-1} \left(\frac{-1}{s+j} \right) \right] \int_0^\infty u^{s+k} f^{(k+1)}(u) du \\
=& (-1) \left[\prod_{j=0}^{k-1} \left(\frac{-1}{-k+j} \right) \right] \int_0^\infty f^{(k+1)}(u) du \\
=& \frac{f^{(k)}(0)}{k!}
\end{aligned}$$

This is the desired result.

Remark. Compare with the analytic continuation of the Γ function and the distribution $x_+^{s-1}/\Gamma(s)$.

18. *Suppose $t \mapsto \gamma(t)$ is a smooth curve in $\mathbb{R}^3$ with non-vanishing torsion (i.e. the vectors $\gamma'(t), \gamma''(t)$, and $\gamma'''(t)$ are linearly independent for each t). Let $d\sigma$ be the measure carried on this curve, given by*

$$\int_{\mathbb{R}^3} f d\sigma = \int_0^1 f(\gamma(t)) |\gamma'(t)| dt$$

Show that $(d\sigma)\hat{}(\xi) = O(|\xi|^{-1/3})$.

We prove this instead for $d\mu = \phi d\sigma$, where $\phi \in C_0^\infty$ such that ϕ restricted to the curve is supported in the interior of the curve. We use a partition of unity (which will be specified later) to write

$$\int e^{-2\pi i x \cdot \xi} d\mu(x) = \sum_n \int_0^1 e^{-2\pi i \gamma(t) \cdot \xi} \psi_n(t) |\gamma'(t)| dt$$

Each of these integrals can be written

$$\int e^{-2\pi i\lambda\gamma(t)\cdot\eta}\psi_n(t)|\gamma'(t)|dt = \int e^{i\lambda\Phi(t)}\psi_n(t)|\gamma'(t)|dt$$

where $\lambda = |\xi|$. This is an oscillatory integral. We know for each t_0, either $\Phi'(t_0), \Phi''(t_0)$, or $\Phi'''(t_0)$ is nonzero from the hypothesis of nonvanishing torsion. This means it is uniformly nonzero in a neighborhood of t_0, so provided our partition of unity is chosen in such a way that on the supports of the ψ_n we have one of Φ', Φ'' or Φ''' bounded away from zero, we can apply a van der Corput estimate. This tells us that $\int e^{i\lambda\Phi(t)}\psi_n(t)|\gamma'(t)|dt = O(\lambda^{-1/k})$, where k is the order of the derivative which we know is nonvanishing on the support of ψ_n. In the worst case we get decay of $O(\lambda^{-1/3}) = O(|\xi|^{-1/3})$.

Remark. This is a prototype of the situation where the submanifold does not satisfy the non-vanishing Gaussian curvature condition, but satisfies a weaker 'finite type' condition. See Chapter 8.3.2 of Stein's *Harmonic Analysis*.

19. *Let S be a smooth hypersurface in $\mathbb{R}^d$ whose curvature vanishes at one point. Then the averages theorem, restriction theorem, etc. may fail as stated. For example, take S to be the curve $x_2 = x_1^k$ in the plane, with k an integer > 2. Then the curvature of S vanishes at the origin only.*

 (a) *Show that in this case the inequality $\|A(f)\|_{L^q} \leq A\|f\|_{L^p}$ cannot hold for $p = 3/2$, $q = 3$.*
 We claim that if A maps $L^p(\mathbb{R}^2)$ to $L^{p'}(\mathbb{R}^2)$, then $\frac{1}{p} - \frac{1}{p'} \leq \frac{1}{k+1}$. As a result, if A maps $L^{\frac{3}{2}}$ to L^3, then $\frac{1}{3} \leq \frac{1}{k+1}$, i.e. $k \leq 2$. Let $f(x) = \chi_{(0,2\varepsilon)}(x_1)\chi_{(0,\varepsilon^k)}(x_2)$. Then

$$\|f\|_{L^p} = c\varepsilon^{\frac{k+1}{p}}.$$

 Also for ε small, if $x_1 \in (0,\varepsilon)$ and $x_2 \in (0,\varepsilon^k)$,

$$Af(x) \geq \int_0^{x_2^{\frac{1}{k}}} f(x_1 + t, x_2 - t^k)\sqrt{1 + k^2t^{2k-2}}dt \geq cx_2^{\frac{1}{k}}$$

 so

$$\|Af\|_{L^{p'}} \geq c\left(\int_0^{\varepsilon}\int_0^{\varepsilon^k}\left(x_2^{\frac{1}{k}}\right)^{p'} dx_2dx_1\right)^{\frac{1}{p'}} = c\varepsilon^{1+\frac{k+1}{p'}}.$$

 For

$$c\varepsilon^{1+\frac{k+1}{p'}} \leq C\varepsilon^{\frac{k+1}{p}}$$

 to hold as $\varepsilon \to 0$ we need $1 + \frac{k+1}{p'} \geq \frac{k+1}{p}$, i.e.

$$\frac{1}{p} - \frac{1}{p'} \leq \frac{1}{k+1}.$$

 (c.f. Chapter 9, 5.21(b) of Stein's *Harmonic Analysis*.)

Remarks. The characteristic functions are typically easy to average along a submanifold, so we have chosen to test the given inequality with such. Note also that implicit in the above solution is the role played by the one-parameter family of *non-isotropic* dilations $(x_1, x_2) \mapsto (\delta x_1, \delta^k x_2)$. These dilations are relevant because they preserve the curve $x_2 = x_1^k$.

(b) *Show that* $\|R(f)\|_{L^2(S)} \le A\|f\|_{L^p(\mathbb{R}^2)}$ *fails for* $p = 6/5$ *in this case.*
Let $\varepsilon \in (0, 1]$ and let χ_ε be the characteristic function of the set

$$\{(\xi_1, \xi_2) : -\varepsilon \le \xi_1 \le \varepsilon, -\varepsilon^k \le \xi_2 \le \varepsilon^k\}$$

and consider $f_\varepsilon = \mathcal{F}^{-1}\chi_\varepsilon$. We have on the one hand

$$\|R(f_\varepsilon)\|_{L^2(S)} = \left(\int_{-\varepsilon}^{\varepsilon} \sqrt{1 + \left(kx_1^{k-1}\right)^2} dx_1\right)^{1/2} \ge \left(\int_{-\varepsilon}^{\varepsilon} dx_1\right)^{1/2} = c\varepsilon^{1/2}.$$

Meanwhile f_ε is given by

$$f_\varepsilon(x) = \int e^{2\pi i x\cdot\xi}\chi_\varepsilon(\xi)d\xi = \int_{-\varepsilon}^{\varepsilon} e^{2\pi i x_1\xi_1}d\xi_1 \int_{-\varepsilon^k}^{\varepsilon^k} e^{2\pi i x_2\xi_2}d\xi_2 = c\frac{\sin(2\pi x_1\varepsilon)}{x_1}\frac{\sin(2\pi x_2\varepsilon^k)}{x_2}.$$

We can then compute

$$\begin{aligned}
\|f_\varepsilon\|_{L^{6/5}(\mathbb{R}^2)} &= \left(\int \frac{|\sin(2\pi x_1\varepsilon)|^{6/5}}{|x_1|^{6/5}}dx_1 \int \frac{|\sin(2\pi x_2\varepsilon^k)|^{6/5}}{|x_2|^{6/5}}dx_2\right)^{5/6} \\
&= \left(\varepsilon^{1/5}\int \frac{|\sin(2\pi x_1)|^{6/5}}{|x_1|^{6/5}}dx_1\right)^{5/6} \left(\varepsilon^{k/5}\int \frac{|\sin(2\pi x_2)|^{6/5}}{|x_2|^{6/5}}dx_2\right)^{5/6} \\
&= \varepsilon^{\frac{1+k}{6}}\|f_1\|_{L^{6/5}}.
\end{aligned}$$

Letting $\varepsilon \to 0$ we see that so long as $k \ge 3$, no inequality of the form $\|R(f)\|_{L^2(S)} \le A\|f\|_{L^{6/5}(\mathbb{R}^2)}$ can be true.

Alternative solution. Let $f(x) = e^{-\pi\varepsilon x_1^2}e^{-\pi\varepsilon^k x_2^2}$. Then $\hat{f}(\xi) = \varepsilon^{-\frac{k+1}{2}}e^{-\frac{\pi\xi_1^2}{\varepsilon}}e^{-\frac{\pi\xi_2^2}{\varepsilon^k}}$. Hence

$$\|Rf\|_{L^2(S)} \ge \varepsilon^{-\frac{k+1}{2}}\left(\int_0^\infty e^{-\frac{2\pi\xi_1^2}{\varepsilon}}e^{-\frac{2\pi\xi_1^{2k}}{\varepsilon^k}}d\xi_1\right)^{\frac{1}{2}} = c\varepsilon^{-\frac{k+1}{2}}\varepsilon^{\frac{1}{4}}.$$

For $\|Rf\|_{L^2(S)} \le C\|f\|_{L^p(\mathbb{R}^2)}$ to hold, we need

$$c\varepsilon^{-\frac{k+1}{2}}\varepsilon^{\frac{1}{4}} \le C\varepsilon^{-\frac{k+1}{2p}}$$

to hold when $\varepsilon \to 0$. Thus we need $-\frac{k+1}{2} + \frac{1}{4} \ge -\frac{k+1}{2p}$, i.e.

$$p \le \frac{2k+2}{2k+1}.$$

If $p = 6/5$ is allowed, then $\frac{6}{5} \le \frac{2k+2}{2k+1}$, i.e. $k \le 2$. Note again the role played by the relevant non-isotropic dilations here. c.f. Chapter 9, 5.15(b) and 5.17(a) of Stein's *Harmonic Analysis*.

20. *Let $u(x,t)$ be the standard solution of the Schrödinger equation $i\partial_t u = \Delta_x u$ with initial condition $u(x,0) = f(x)$ given by*

$$u(x,t) = \int_{\mathbb{R}^d} e^{4\pi^2 it|\xi|^2} e^{2\pi i x\cdot\xi} \hat{f}(\xi) d\xi$$

Prove

(a) $\sup_x |u(x,t)| \leq ct^{-d/2}\|f\|_1$.

Let $g(x) = (-4\pi it)^{-d/2} e^{|x|^2/4it}$ (with the square root chosen in the right half-plane). Then $\hat{g}(\xi) = e^{4\pi^2 it|\xi|^2}$ (see the appendix). Now suppose $f \in C_0^\infty(\mathbb{R}^d)$. We write

$$u(x,t) = \int_{\mathbb{R}^d} e^{2\pi ix\cdot\xi} \hat{f}\hat{g} d\xi = f * g(x) = \int_{\mathbb{R}^d} f(x-y)(-4\pi it)^{-d/2} e^{|y|^2/4it} dy.$$

Hence $|u(x,t)| \leq ct^{-d/2}\|f\|_1$ for all x. A density argument allows us to extend the result to all $f \in L^1$.

(b) *When $d = 1$, $\sup_x \int_0^\infty |\partial_x u(x,t)|^2 dt \leq c\|f\|^2_{L^2_{1/2}}$.*

We first compute $\partial_x u(x,t) = \int e^{4\pi^2 it\xi^2}(2\pi i\xi) e^{2\pi ix\xi} \hat{f}(\xi) d\xi$. We then divide the integral into two pieces and make the change of variable $\eta = \xi^2$ in each:

$$\int_0^\infty e^{4\pi^2 it\xi^2}(2\pi i\xi) e^{2\pi ix\xi}\hat{f}(\xi) d\xi = \int_0^\infty e^{4\pi^2 it\eta}(\pi i) e^{2\pi ix\sqrt{\eta}} \hat{f}(\sqrt{\eta}) d\eta$$

$$\int_{-\infty}^0 e^{4\pi^2 it\xi^2}(2\pi i\xi) e^{2\pi ix\xi}\hat{f}(\xi) d\xi = \int_\infty^0 e^{4\pi^2 it\eta}(\pi i) e^{-2\pi ix\sqrt{\eta}} \hat{f}(-\sqrt{\eta}) d\eta$$

Combining these two and using another change of variable to rescale, we find

$$\begin{aligned}\partial_x u(x,t) &= \frac{1}{2}\int_0^\infty e^{2\pi it\eta}(i)\left(e^{ix\sqrt{2\pi\eta}}\hat{f}(\sqrt{\eta/2\pi}) - e^{-ix\sqrt{2\pi\eta}}\hat{f}(-\sqrt{\eta/2\pi})\right) d\eta \\ &\stackrel{\text{def}}{=} \int_{-\infty}^\infty e^{2\pi it\eta}\varphi(x,\eta) d\eta.\end{aligned}$$

Here φ is defined in such a way that $\varphi(x,\eta) = 0$ for $\eta < 0$. Applying now Plancherel's theorem we see that

$$\begin{aligned}\int_{-\infty}^\infty |\partial_x u(x,t)|^2 dt &= \int_{-\infty}^\infty |\varphi(x,\eta)|^2 d\eta \\ &\leq \frac{1}{4}\int_0^\infty |\hat{f}(\sqrt{\eta/2\pi})|^2 + |\hat{f}(-\sqrt{\eta/2\pi})|^2 d\eta \\ &= \pi\int_0^\infty (|\hat{f}(\xi)|^2 + |\hat{f}(-\xi)|^2)\xi d\xi \\ &= \pi\int_{-\infty}^\infty |\hat{f}(\xi)|^2 |\xi| d\xi \leq \pi\|f\|^2_{L^2_{1/2}}\end{aligned}$$

From this the desired result $\int_0^\infty |\partial_x u(x,t)|^2 dt \leq c\|f\|^2_{L^2_{1/2}}$ follows.

Appendix

Let $x \in \mathbb{R}$ and $f(x) = e^{-\pi\delta x^2}$ for $\delta \neq 0, \Re(\delta) \geq 0$. We compute $\hat{f}(\xi)$ as follows: Observe that f satisfies the following differential equation

$$(\frac{d}{dx} + 2\pi\delta x)f(x) = 0$$

Taking the Fourier transform of both sides we see that

$$(2\pi i\xi + i\delta\frac{d}{d\xi})\hat{f}(\xi) = 0$$

We multiply by the integrating factor $e^{\pi|\xi|^2/\delta}$ (our distributions may no longer be tempered at this point) to get

$$\frac{d}{d\xi}(e^{\pi\xi^2/\delta}\hat{f}(\xi)) = 0$$

But a distribution of zero derivative is constant, so that

$$\hat{f}(\xi) = Ce^{-\pi\xi^2/\delta}$$

Now $C = \hat{f}(0) = \int f(x)dx$. If δ is real, this can be computed by a change of variables to be $\delta^{-1/2}$. If δ is complex, after multiplying and dividing by the square root of δ which lies in the right half plane, we write the integral as a contour integral over the line Γ given by $\{z = \delta(x) : x \in \mathbb{R}\}$:

$$\int_{-\infty}^{\infty} e^{-\pi\delta x^2}dx = \delta^{-1/2}\int_{\Gamma} e^{-\pi z^2}dz$$

Now a deformation of contour shows that this last integral is 1. In the $d-$dimensional case this gives us

$$(e^{-\pi\delta|x|^2})^{\wedge} = \delta^{-d/2}e^{-\pi|\xi|^2/\delta}$$

with the square root of δ chosen in the right half plane.