Selected problems from week 2 of the Princeton analysis/geometry summer program
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2. (The Cousin I Problem for Domains in $\mathbb{C}$)

   Part (1) is trivial; we will deal with part (2). Let $\{\varphi_j\}$ be a partition of unity relative to the cover $\{U_j\}$. On each U_j, define a smooth function g_j by

   $$g_j = \sum_i \varphi_i h_{j,i}.$$

   To see that g_j is well-defined, first note that since φ_i is supported inside U_i, the function $\varphi_i h_{j,i}$, which is *a priori* only defined on $U_i \cap U_j$, can be extended by 0 on $U_j \setminus (U_i \cap U_j)$ in a smooth manner. Then, since the cover $\{U_i\}$ is locally finite, in a neighborhood of any point p of U_j the sum defining g_j is a finite sum of smooth functions and thus is itself smooth near p.

   These functions g_j form a cochain with respect to the cover $\{U_j\}$. Computing the coboundary, we see that $g_j - g_k = \sum_i \varphi_i(h_{j,i} - h_{k,i}) = (\sum_i \varphi_i)(h_{j,k}) = h_{j,k}$ on $U_j \cap U_k$, since $\sum_i \varphi_i = 1$. The second equality used the cocycle conditions on h, giving $h_{j,i} - h_{k,i} = h_{j,i} + h_{i,k} = -h_{k,j} = h_{j,k}$.

   This means that we have split the cocycle $\{h_{j,k}\}$ over the sheaf of smooth functions on Ω; we want to split it over the sheaf of holomorphic functions. To do this, we want to add a "correcting" cochain $\{r_j\}$ to $\{g_j\}$ such that the following two conditions hold: $\overline{\partial} r_j = \overline{\partial} g_j$ on U_j, and $\delta\{r_j\} = 0$, where δ is the coboundary operator. If such r_j exist, then $\{f_j\} := \{g_j - r_j\}$ is a cochain of holomorphic functions whose coboundary is still $\{h_{j,k}\}$, and we will be done. Thus, we have reduced the problem to the existence of these r_j.

   Note that the coboundary condition on $\{r_j\}$ amounts to saying that $r_j = r_k$ on $U_j \cap U_k$, or equivalently that the functions r_j are the restrictions of a globally defined smooth function r on Ω. Similarly, there is a globally defined 1-form ω on Ω such that the forms $\overline{\partial} g_j$ are just the restrictions of ω to U_j; this is because $\overline{\partial} g_j - \overline{\partial} g_k = \overline{\partial} h_{j,k} = 0$ on $U_j \cap U_k$. By Theorem 1.1 in Prof. Nagel's notes, we can solve the equation $\overline{\partial} r = \omega$ on Ω. Then if $r_j := r|_{U_j}$, we have $\overline{\partial} r_j = \omega|_{U_j} = \overline{\partial} g_j$ as desired, finishing the proof.

6. (Solving $\overline{\partial}$ with compact support)

   In the forward direction, suppose such a ψ exists, and let R be large enough that the support of ψ is contained in $|z| < R$ (this ensures the support of φ is contained in this set too). Let $n \geq 0$; then $\iint_{|z|\leq R} \varphi(z) z^n dz \wedge d\overline{z} = \iint_{|z|\leq R} \frac{\partial \psi}{\partial \overline{z}} z^n dz \wedge d\overline{z} = \iint_{|z|\leq R} d(\psi(z) z^n dz)$. By Stokes' theorem, this equals $\int_{|z|=R} \psi(z) z^n dz = 0$ since the circle $|z| = R$ lies outside the support of ψ.

   Conversely, suppose this integral vanishes for all $n \geq 0$. We at least know how to solve $\frac{\partial \psi}{\partial \overline{z}} = \varphi$ in the smooth (non-compactly supported) setting; let ψ be such a solution. We want to find an entire function h which agrees with ψ outside some compact set; then $\psi - h$ still solves our equation but is compactly supported.

   Choose R large enough that the support of φ is contained in the disc $|z| \leq R$. The same chain of equalities used above to prove the forward direction still hold, and we have $0 = \int_{|z|=R} \psi(z) z^n dz$. To compute this integral, we can use the parametrization $z = Re^{2\pi i t}, 0 \leq t \leq 1$ to get the identity $0 = -2\pi i R^{n+1} \int_0^1 \psi(Re^{2\pi i t}) e^{2\pi i (n+1) t} dt$. Let $f_R(t) = \psi(Re^{2\pi i t})$; then this integral is $-2\pi i R^{n+1}$ times the $(-n-1)^{st}$ Fourier coefficient of f_R. $-2\pi i R^{n+1}$ is nonzero, so we see that all the negative Fourier coefficients of f_R vanish.

   The periodic function f_R is smooth, so by a general theorem from Fourier analysis, f_R equals its Fourier series everywhere and this series converges absolutely. Thus, if a_k is the k^{th} Fourier coefficient of f_R, we have $f_R(t) = \sum_{k=0}^{\infty} a_k e^{2\pi i k t}$, and $\sum |a_k| < \infty$. We will define a power series related to this Fourier series: let $g(z) = \sum_{k=0}^{\infty} a_k (\frac{z}{R})^k$. The absolute convergence of $\sum |a_k|$ and the Weierstrass M-test imply that g converges uniformly on the closed disc $|z| \leq R$; thus g is continuous on this disc (and holomorphic inside it). For any $z = Re^{2\pi it}$ on the circle $|z| = R$, we have $g(z) = \sum a_k e^{2\pi ikt} = f_R(t) = \psi(Re^{2\pi it}) = \psi(z)$.

Let p be any point on the circle $|z| = R$ and consider the function $g - \psi$ near p. Let U be a small neighborhood of p which is symmetric with respect to $|z| = R$. We know that $g - \psi$ is holomorphic on $U \cap \{|z| < R\}$, continuous on $U \cap \{|z| \leq R\}$, and zero (in particular, real-valued) on $U \cap \{|z| = R\}$. By the Schwartz reflection principle, $g - \psi$ can be analytically continued to all of U. Since ψ is holomorphic on U already, g can be analytically continued to all of U. The functions g and ψ are now holomorphic on domains which cover all of $\mathbb{C}$ and whose intersection contains a circle segment on which $g = \psi$. Therefore, $g = \psi$ on the whole intersection of their domains of analyticity, so they patch together to give an entire function. If we let h be this entire function, we are done.

To prove the final claim, let φ be any bump function with integral 1 on $\mathbb{C}$, and let $n = 0$. Then $\iint \varphi(z) dz \wedge d\bar{z} \neq 0$, so the equation $\frac{\partial \psi}{\partial \bar{z}} = \varphi$ does not admit a smooth solution ψ with compact support.