A LINEAR MULTIPLICATIVE MAP FROM A UNITAL BANACH ALGEBRA INTO $\mathbb{C}$ IS CONTINUOUS

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First we give a preliminary definition/lemma. If B is a normed linear space and V a closed subspace then for $[b] \in B/V$ (where $[b]$ denotes the equivalence class of b) we define $||[b]||_{B/V} = \inf_{v\in V} ||b - v||_B$. It is easy to check that $||\cdot||_{B/V}$ is well defined and satisfies all of the conditions to be a norm.

Lemma 0.1. *The map $\varphi : B \to B/V$ given by $b \to [b]$ is continuous.*

Proof. This is immediate since

$$||\varphi(b)||_{B/V} = \inf_{v\in V} ||b - v||_B \leq ||b||_B$$

□

Lemma 0.2. *If f is a non zero map of a normed linear space B into $\mathbb{C}$, then $\ker(f)$ has codimension 1. In other words, $B \cong \ker(f) \oplus \mathbb{C}$.*

Proof. Choose x such that $f(x) \neq 0$. Now let $y \in B$. We have

$$y = \left(y - \frac{f(y)}{f(x)}x\right) + \frac{f(y)}{f(x)}x$$

Thus every y can be written as $y = \lambda + cx$ for some scalar c and some $f(\lambda) = 0$. Since

$$\lambda_1 + c_1x = \lambda_2 + c_2x \Rightarrow \lambda_1 - \lambda_2 = (c_2 - c_1)x \Rightarrow (c_2 - c_1)f(x) = 0 \Rightarrow c_2 = c_1$$

we see that this representation is unique and the lemma follows. □

Lemma 0.3. *Suppose that f is a linear map from a normed linear space B into $\mathbb{C}$. Then f is continuous if and only $\ker(f)$ is closed.*

Proof. If f is continuous then we clearly have $\ker(f) = f^{-1}(0)$ is closed. Now assume that $\ker(f)$ is closed. Define $\hat{f} : B/\ker(f) \to \mathbb{C}$ by setting $\hat{f}([b]) = f(b)$. This is clearly well defined and linear. From the previous lemma we know that $B/\ker(f)$ has dimension 1. Since linear maps on finite dimensional subspaces are continuous we have that $\hat{f}$ is continuous. Since $f = \hat{f} \circ \varphi$ we see that f is also continuous. □

Lemma 0.4. *If B is a unital Banach algebra and we choose $b \in B$ with $||b||_B < 1$, then $1 - b$ is invertible.*

Proof. Motivated by geometric series we notice the following

$$||\sum_{k=N}^{M} b^k||_B \leq \sum_{k=N}^{M} ||b||_B^k \to 0 \text{ as } N, M \to \infty$$

Date: August 6, 2009.

Thus the sequence $\sum_{k=0}^{\infty} b^k$ is convergent is B. We have

$$(1-b)(\sum_{k=0}^{\infty} b^k) = \lim_{N\to\infty}(1-b)\sum_{k=0}^{N} b^k = \lim_{N\to\infty} 1 - b^N = 1$$

where we have used the fact that $b^N \to 0$. Similarly

$$(\sum_{k=0}^{\infty} b^k)(1-b) = 1$$

Thus $(1-b)^{-1} = \sum_{k=0}^{\infty} b^k$. □

Lemma 0.5. *If I is an ideal of codimension 1 in a unital Banach algebra B then I is closed.*

Proof. Every element $b \in B$ can be uniquely written as $b = \lambda + z$ where $\lambda \in I$ and z is in the span of x for fixed x not in I. We have $b \in I$ if and only $z = 0$. Choose some $y = \lambda + cx$ not in I, i.e. with $c \neq 0$. Then $\lambda \in I$ and we have $x = (1/c)(y - \lambda)$. This implies that if $\hat{I}$ is an ideal with $I \subset \hat{I}$ then either $\hat{I} = I$ or $\hat{I} = B$. Let $\bar{I}$ denote the closure of I. Since multiplication and addition are continuous maps it is easy to see that $\bar{I}$ is an ideal. Thus, if we show that $\bar{I} \neq B$, we will have shown that I is closed. Since I has codimension 1 it is proper and hence $1 \notin I$. From a lemma above we showed that the ball of radius 1 about 1 is also not contained in I. Thus the closure of I will not contain 1. This shows that I is closed. □

Theorem 0.6. *Any multiplicative and linear map from a unital Banach algebra into $\mathbb{C}$ is automatically continuous.*

Proof. Let f be the multiplicative and linear map. If $f = 0$ the statement is trivial so assume otherwise. It is immediately clear that $\ker(f)$ is an ideal. Via lemma 2 we know that it has codimension 1. Then from the above lemma we have that $\ker(f)$ is closed. Finally we can apply the third lemma to see that f must be continuous. □