2. (2) Let $\{U_j\}$ be a locally finite open cover of an open set $\Omega \subseteq \mathbb{C}$. Let $h_{j,k}$ be holomorphic functions on $U_j \cap U_k$ such that

$$h_{j,k} + h_{k,j} = 0 \quad \text{on } U_j \cap U_k$$

$$h_{j,k} + h_{k,l} + h_{l,j} = 0 \quad \text{on } U_j \cap U_k \cap U_l.$$

Show that there exists holomorphic functions f_j on U_j for each j, such that

$$h_{j,k} = f_j - f_k \quad \text{on } U_j \cap U_k$$

for all j, k.

6. We proved the Hartog's extension theorem by first showing that if g is a $(0,1)$- form in $\mathbb{C}^n$, $n \geq 2$, with coefficients which are infinitely differentiable with compact support, and if $\overline{\partial} g = 0$, then there exists $u \in C_c^\infty(\mathbb{C}^n)$ with $\overline{\partial} u = g$. We want to show that this result is false when $n = 1$.

Problem. Let $\varphi \in C_c^\infty(\mathbb{C})$ be a smooth function with compact support. Prove that there exists $\psi \in C_c^\infty(\mathbb{C})$ such that

$$\frac{\partial \psi}{\partial \overline{z}} = \varphi$$

if and only if for every non-negative integer n we have

$$\int_{\mathbb{C}} \varphi(z) z^n dz \wedge d\overline{z} = 0.$$

As a consequence, prove that there exists $\varphi \in C_c^\infty(\mathbb{C})$ such that there does not exist a function $\psi \in C_c^\infty(\mathbb{C})$ with $\frac{\partial \psi}{\partial \overline{z}} = \varphi$.