AN INTRODUCTION TO SEVERAL COMPLEX VARIABLES (1ST DRAFT)

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In this note we provide some background for the inhomogeneous Cauchy-Riemann equation and the $\overline{\partial}$-Neumann problem on domains in $\mathbb{C}^{n+1}$, $n \geq 1$. The emphasis will be on the existence and the regularity of weak solutions.

1. The inhomogeneous Cauchy-Riemann equation

Let Ω be a bounded domain in $\mathbb{C}^{n+1}$ with smooth boundary, $n \geq 1$. We shall use the standard Euclidean coordinates on Ω: $z = (z_1, \dots, z_{n+1})$,

$$z_j := x_j + iy_j, \quad j = 1, \dots, n+1.$$

We have the $(1,0)$ forms

$$dz_j := dx_j + idy_j$$

and the $(0,1)$ forms

$$d\overline{z}_j := dx_j - idy_j.$$

We also have the holomorphic vector fields

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)$$

and the anti-holomorphic vector fields

$$\frac{\partial}{\partial \overline{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

(The coefficients were chosen so that the dz_j and $d\overline{z}_j$'s are dual to the $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial \overline{z}_j}$'s.) A $(0,q)$ form is an alternating tensor product of q $(0,1)$ forms, and a $(0,0)$ form is just a function. A basis of $(0,q)$ forms is given by $\{d\overline{z}_I\}$ where I runs over all strictly increasing multi-indices of length q and $d\overline{z}_I := d\overline{z}_{i_1} \wedge \cdots \wedge d\overline{z}_{i_q}$ if $I = (i_1, \dots, i_q)$. The (complex) vector space of $(0,q)$ forms at a point carries a Hermitian inner product $\langle \cdot, \cdot \rangle$ that makes the preceding basis an orthogonal one at every point. The vector space of all $(0,q)$ forms on Ω with L^2 coefficients, denoted $L^2_{(0,q)}(\Omega)$, is then equipped with a Hermitian inner product

$$(u, v) = \int_\Omega \langle u, v \rangle dz$$

which makes it a Hilbert space, where dz is the standard Euclidean measure on $\mathbb{C}^{n+1}$.

We shall now define the *Cauchy-Riemann operator* $\overline{\partial}$. In distribution, it is given by

$$\overline{\partial} u := \sum_I \sum_{j=1}^{n+1} \frac{\partial u_I}{\partial \overline{z}_j} d\overline{z}_j \wedge d\overline{z}_I \qquad \text{if } u = \sum_I u_I d\overline{z}_I.$$

Hereafter sums like $\sum_I$ will always mean sums over strictly increasing multi-indices. It sends $(0,q)$ forms to $(0,q+1)$ forms. In particular, a function u is holomorphic if and only

Date: 3 August 2009.

if $\overline{\partial} u = 0$. Since we shall be working with the Hilbert space $L^2_{(0,q)}(\Omega)$ in a moment, from now on, however, unless otherwise specified, we shall take $\overline{\partial}$ to be the (unbounded) linear operator

$$\overline{\partial} \colon L^2_{(0,q)}(\Omega) \to L^2_{(0,q+1)}(\Omega)$$

with domain

$$\text{Dom}(\overline{\partial}) = \{u \in L^2_{(0,q)}(\Omega) \colon \text{the distributional } \overline{\partial} u \in L^2_{(0,q+1)}(\Omega)\}$$

so that the Hilbert space operator $\overline{\partial}$ agrees with the distributional $\overline{\partial}$ whenever the former is defined.

Another often useful reformulation of the definition of $\overline{\partial}$ is the following. If $\overline{Z}_1, \dots, \overline{Z}_{n+1}$ is a basis of anti-holomorphic vector fields at each point of Ω, and $\overline{\omega}_1, \dots, \overline{\omega}_{n+1}$ is the dual basis of $(0,1)$ forms to $\overline{Z}_1, \dots, \overline{Z}_{n+1}$, then one can define the distributional $\overline{\partial}$ by requiring

$$\overline{\partial} u = \sum_{j=1}^{n+1} (\overline{Z}_j u) \overline{\omega}_j \qquad \text{for all functions } u$$

and that

$$\overline{\partial}(u \wedge v) = (\overline{\partial} u) \wedge v + (-1)^q u \wedge (\overline{\partial} v) \qquad \text{for all forms } u \text{ and } v$$

whenever u is a $(0,q)$ form. We can then define the Hilbert space operator $\overline{\partial}$ as above, and again from now on the symbol $\overline{\partial}$ shall refer to the Hilbert space operator.

One important property of the $\overline{\partial}$ operator is that it forms a *complex*: in other words, $\text{Range}(\overline{\partial}) \subseteq \text{Dom}(\overline{\partial})$, and

$$\overline{\partial} \circ \overline{\partial} = 0.$$

One fundamental question in several complex variables is to solve the following *inhomogeneous Cauchy-Riemann equation* for $u \in L^2_{(0,q)}(\Omega)$:

$$\overline{\partial} u = f, \quad u \perp \text{kernel of } \overline{\partial}. \tag{1}$$

In other words, we want to solve the above equation weakly for $u \in L^2_{(0,q)}(\Omega)$, assuming $f \in L^2_{(0,q+1)}(\Omega)$ is given. Since $\overline{\partial}$ forms a complex, this equation can only have a solution when the compatibility condition $\overline{\partial} f = 0$ is satisfied, which we shall always assume from now on. Another way of viewing this is that this system of equations is over-determined, and some compatibility conditions must be imposed on the given data. The orthogonality condition on u was made to ensure that the solution is unique (if it exists). The existence of weak solution to this inhomogeneous Cauchy-Riemann equation is our main concern in this section.

1.1. Pseudoconvexity.

It turns out that solutions to the inhomogeneous Cauchy-Riemann equation may or may not exist in such a general formulation. To ensure the existence of solutions, one needs to impose some geometric condition on the boundary of Ω. This is usually formulated in terms of *pseudoconvexity*, a concept to which we now turn.

Again let Ω be a bounded domain in $\mathbb{C}^{n+1}$ with smooth boundary. First we should introduce the important concept of *holomorphic tangent vectors* on the boundary $\partial\Omega$ of Ω. Remember that a vector at $z \in \mathbb{C}^{n+1}$ is said to be holomorphic if it is a complex linear combination of

$$\left.\frac{\partial}{\partial z_j}\right|_z, \quad j = 1, \dots, n+1;$$

the set of all such is denoted as $T^{(1,0)}_z(\mathbb{C}^{n+1})$, and is a complex vector space of dimension $n+1$. Now for each $z \in \partial\Omega$, let $\mathbb{C}T_z(\partial\Omega) := T_z(\partial\Omega) \otimes \mathbb{C}$ be the complexified tangent space at z to $\partial\Omega$ (in other words, the space of all vectors at z with complex coefficients whose real and imaginary parts are both tangent to $\partial\Omega$), and let $T_z^{(1,0)}(\partial\Omega)$ be the intersection of this complexified tangent space with $T_z^{(1,0)}(\mathbb{C}^{n+1})$. $T_z^{(1,0)}(\partial\Omega)$ is then a complex vector space of dimension n, and elements of this space will be called *holomorphic tangent vectors* to $\partial\Omega$ at z.

Fix now a smooth defining function ρ for Ω. This means that

$$\Omega = \{z \in \mathbb{C}^{n+1} \colon \rho(z) < 0\},$$

with $|d\rho| \neq 0$ at every point on $\partial\Omega$. If $z \in \partial\Omega$, the matrix

$$\left(\frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(z) \right)_{1 \leq j,k \leq n+1}$$

defines a Hermitian form on $T_z^{(1,0)}(\mathbb{C}^{n+1})$, and the *Levi form* L_z at z is defined to be the *restriction* of this Hermitian form to $T_z^{(1,0)}(\partial\Omega)$. More explicitly, if $Z = \sum_{j=1}^{n+1} a_j \frac{\partial}{\partial z_j}$ and $W = \sum_{j=1}^{n+1} b_j \frac{\partial}{\partial z_j}$ are tangent to $\partial\Omega$ at z, then

$$L_z(Z, W) := \sum_{j,k=1}^{n+1} a_j \overline{b_k} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(z).$$

Clearly the Levi form is also a Hermitian form, and if it is non-negative definite at some $z \in \partial\Omega$, we say that $\partial\Omega$ is *pseudoconvex* at z; *strongly pseudoconvex* if it is positive definite. If $\partial\Omega$ is pseudoconvex at every point, then we say Ω is *pseudoconvex*; similarly for strongly pseudoconvexity.

Note that while the definition of the Levi form above depends on the choice of the defining function ρ, whether or not a domain is pseudoconvex (or strongly pseudoconvex) is independent of the choice of the defining function.

The name pseudoconvex was chosen because every (smooth) convex domain is pseudoconvex. c.f. Exercise 4.

There is actually a notion of pseudoconvexity when the domain is not smooth, but we shall not discuss that.

1.2. **Hilbert space reformulation.** In the following, we shall prove the existence of weak solutions to the inhomogeneous Cauchy-Riemann equation (1) on all bounded pseudoconvex domains with smooth boundaries. To do so, we shall need the theory of closed operators on Hilbert spaces. Recall that a densely defined linear operator $T \colon H_1 \to H_2$ between two Hilbert spaces is said to be *closed* if its graph is closed in $H_1 \times H_2$. It is easy to check that the operator $\overline{\partial} \colon L^2_{(0,q)}(\Omega) \to L^2_{(0,q+1)}(\Omega)$ we defined is a closed operator. At this point it is convenient to introduce the Hilbert space adjoint of $\overline{\partial}$, denoted by

$$\overline{\partial}^* \colon L^2_{(0,q+1)}(\Omega) \to L^2_{(0,q)}(\Omega).$$

Then $\overline{\partial}^*$ is also densely defined, linear and closed.

Now one can check that the orthogonal complement of (the closure of) the range of $\overline{\partial}$ in $L^2_{(0,q+1)}(\Omega)$ is the kernel of $\overline{\partial}^*$. Hence

$$L^2_{(0,q+1)}(\Omega) = \text{Kernel}(\overline{\partial}^*) \oplus \overline{\text{Range}(\overline{\partial})} \tag{2}$$

where $\oplus$ denotes an orthogonal direct sum. Going back to the inhomogeneous Cauchy-Riemann equation, suppose $f \in L^2_{(0,q+1)}(\Omega)$ is given with $\overline{\partial} f = 0$. We want to find $u \in L^2_{(0,q)}(\Omega) \cap \text{Dom}(\overline{\partial})$ such that $\overline{\partial} u = f$. If we can find such u, then by orthogonally projecting onto the orthogonal complement of the kernel of $\overline{\partial}$ (which is a closed subspace of $L^2_{(0,q)}(\Omega)$), one can easily obtain a solution of the inhomogeneous Cauchy-Riemann equation (1). To find such an u amounts to showing that $f \in \text{Range}(\overline{\partial})$, to which we now turn.

Suppose $f \in L^2_{(0,q+1)}(\Omega)$ with $\overline{\partial} f = 0$. Using (2) we decompose

$$f = f_1 + f_2, \qquad f_1 \in \text{Kernel}(\overline{\partial}^*), \qquad f_2 \in \overline{\text{Range}(\overline{\partial})}.$$

Note that as a result $f_2 \in \text{Kernel}(\overline{\partial})$. Since we already have $f \in \text{Kernel}(\overline{\partial})$, we have $f_1 \in \text{Kernel}(\overline{\partial})$ as well. If we could show that

(i) $\text{Kernel}(\overline{\partial}) \cap \text{Kernel}(\overline{\partial}^*) = 0$ on $L^2_{(0,q+1)}(\Omega)$, and
(ii) $\text{Range}(\overline{\partial})$ is closed in $L^2_{(0,q+1)}(\Omega)$,

then $f_1 = 0$, hence $f = f_2 \in \text{Range}(\overline{\partial})$ as desired. Hence we are reduced to showing (i) and (ii). This can be accomplished in one stroke if we could show the following *basic estimate* $(q \geq 0)$:

$$\|f\|_{L^2} \leq C(\|\overline{\partial} f\|_{L^2} + \|\overline{\partial}^* f\|_{L^2}), \qquad f \in L^2_{(0,q+1)}(\Omega) \cap \text{Dom}(\overline{\partial}) \cap \text{Dom}(\overline{\partial}^*). \tag{3}$$

In fact it is clear that (3) implies (i), and from (3) it follows that

$$\|f\|_{L^2} \leq C\|\overline{\partial}^* f\|_{L^2}$$

for all $f \in L^2_{(0,q+1)}(\Omega) \cap \text{Dom}(\overline{\partial}^*)$ orthogonal to the kernel of $\overline{\partial}^*$, so the range of $\overline{\partial}^*$ is closed in $L^2_{(0,q)}(\Omega)$, and (ii) follows. (c.f. Exercise 7.) Hence it is tempted to prove the basic estimate on domains Ω that are pseudoconvex.

1.3. **The basic estimate.** It turns out that while the basic estimate for $(0, q+1)$ forms can be established relatively easily on strongly pseudoconvex domains (or a slightly bigger class of domains that satisfies the condition called $Z(q+1)$), in general one should proceed differently. (c.f. however Exercise 13.) Instead of working with the Euclidean measure dz on Ω, we shall work with the weighted Euclidean measure $e^{-\phi(z)}dz$, where $\phi(z)$ is a smooth function on $\overline{\Omega}$. Note that $L^2_{(0,q)}(\Omega)$ is still a Hilbert space under the twisted inner product

$$(u, v)_\phi := \int_\Omega \langle u, v\rangle e^{-\phi} dz, \tag{4}$$

and that the new Hilbert space norm, which we denote as $\|\cdot\|_\phi$, is comparable to the old L^2 norm. Also, $\overline{\partial}\colon L^2_{(0,q)}(\Omega) \to L^2_{(0,q+1)}(\Omega)$ is still closed under this new inner product, and we can still define the Hilbert space adjoint of $\overline{\partial}\colon L^2_{(0,q)}(\Omega) \to L^2_{(0,q+1)}(\Omega)$ under this new inner product, which we denote by

$$\overline{\partial}^*_\phi\colon L^2_{(0,q+1)}(\Omega) \to L^2_{(0,q)}(\Omega).$$

The upshot is that we can run all the previous argument, and show that if for some ϕ, the *weighted basic estimate*

$$\|f\|_\phi \leq C(\|\overline{\partial} f\|_\phi + \|\overline{\partial}^*_\phi f\|_\phi), \qquad f \in L^2_{(0,q+1)}(\Omega) \cap \text{Dom}(\overline{\partial}) \cap \text{Dom}(\overline{\partial}^*_\phi) \tag{5}$$

holds for all $q \geq 0$ (where $\|\cdot\|_\phi$ denotes the L^2 norm under the measure $e^{-\phi}dz$), then the inhomogeneous Cauchy-Riemann equation (1) can be solved weakly.

So suppose now that Ω is a bounded pseudoconvex domain with smooth boundary. We shall content ourselves to proving an *a priori* version of the weighted basic estimate (5) for a suitable ϕ. In other words, we shall just prove the estimate for $(0, q+1)$ forms $f \in \text{Dom}(\overline{\partial}) \cap \text{Dom}(\overline{\partial}^*_\phi)$ that are smooth up to the boundary. The real estimate for all

$f \in L^2_{(0,q+1)}(\Omega) \cap \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*_\phi)$ can then be shown using a density argument, which we shall not give.

To prove the desired a priori estimate, first note that if f is smooth up to the boundary, then it is always in $\mathrm{Dom}(\overline{\partial})$; and it is in $\mathrm{Dom}(\overline{\partial}^*_\phi)$ if and only if

$$\sum_{j=1}^{n+1} f_{jJ'} \frac{\partial \rho}{\partial z_j} = 0 \quad \text{on } \partial\Omega \tag{6}$$

for all strictly increasing multi-indices J' of length q, where ρ is a defining function of Ω with the additional property that $|\nabla \rho| = 1$ on the boundary (so that $\nabla\rho$ is the outward *unit* normal). Hereafter we shall write $f_{jJ'} = \varepsilon^{J,jJ'} f_J$ if $j \notin J'$, where J is the strictly increasing multi-index that is a permutation of $(j, J'_1, \dots, J'_q)$, and $\varepsilon^{J,jJ'}$ is the sign of this permutation. We shall also let $f_{jJ'} = 0$ if $j \in J'$. To prove (6), let $f \in \mathrm{Dom}(\overline{\partial}^*_\phi)$ be smooth up to boundary. Then there exists $g \in L^2_{(0,q)}(\Omega)$ such that

$$(f, \overline{\partial} h)_\phi = (g, h)_\phi$$

for all $h \in L^2_{(0,q)}(\Omega) \cap \mathrm{Dom}(\overline{\partial})$. Now remember the integration by parts formula:

$$\int_\Omega \frac{\partial u}{\partial z_j} v dz = -\int_\Omega u \frac{\partial v}{\partial z_j} dz + \int_{\partial\Omega} uv \frac{\partial \rho}{\partial z_j} d\sigma.$$

Hence

$$\begin{aligned}(f, \overline{\partial} h)_\phi &= \int_\Omega \sum_{J'} \sum_{j=1}^{n+1} f_{jJ'} \frac{\partial \overline{h_{J'}}}{\partial z_j} e^{-\phi} dz \\ &= -\left(\sum_{j=1}^{n+1} \frac{\partial (f_{jJ'} e^{-\phi})}{\partial z_j} e^{\phi} d\overline{z}_{J'}, h \right)_\phi + \int_{\partial\Omega} \sum_{J'} \sum_j f_{jJ'} \overline{h_{J'}} \frac{\partial \rho}{\partial z_j} d\sigma\end{aligned}$$

for all $(0, q)$ forms h that are smooth up to the boundary. In particular this has to be true for all h that are smooth and has compact support, so

$$\overline{\partial}^*_\phi f = -\sum_{J'} \sum_{j \notin J'} \frac{\partial (f_{jJ'} e^{-\phi})}{\partial z_j} e^{\phi} d\overline{z}_{J'}, \tag{7}$$

and it follows that the boundary integral in the previous identity has to vanish for all h that are smooth up to boundary. This proves (6), and the converse is easier. (Note incidentally that the boundary condition (6) does not depend on the choice of the weight ϕ, although $\overline{\partial}^*_\phi$ certainly does.)

Now to prove the a priori weighted basic estimate, let $f \in \mathrm{Dom}(\overline{\partial}^*_\phi)$ be a $(0, q+1)$ form smooth up to the boundary ($q \geq 0$). Observe that

$$\|\overline{\partial} f\|^2_\phi = \int_\Omega \sum_{J,K} \sum_{j,k=1}^{n+1} \varepsilon^{jJ,kK} \frac{\partial f_J}{\partial \overline{z}_j} \overline{\frac{\partial f_K}{\partial \overline{z}_k}} e^{-\phi} dz.$$

Here $\varepsilon^{jJ,kK}$ is 0 unless $j \notin J$, $k \notin K$ and the ordered pair $jJ := (j, J_1, \dots, J_{q+1})$ is a permutation of $kK := (k, K_1, \dots, K_{q+1})$, in which case it is the sign of this permutation. When the latter happens and $j \neq k$, we can choose a (strictly increasing) multiindex J' of length q such that J is a permutation of kJ' and K is a permutation of jJ', where the permutations have opposite signs. Hence

$$\|\overline{\partial} f\|^2_\phi = \int_\Omega \sum_J \sum_{j \notin J} \left| \frac{\partial f_J}{\partial \overline{z}_j} \right|^2 e^{-\phi} dz - \int_\Omega \sum_{J'} \sum_{j \neq k} \frac{\partial f_{jJ'}}{\partial \overline{z}_k} \overline{\frac{\partial f_{kJ'}}{\partial \overline{z}_j}} e^{-\phi} dz.$$

If we now allow in the second sum also the terms where $j = k$, then we commit an error that can be absorbed into the first term, which gives

$$\|\overline{\partial} f\|_\phi^2 = \int_\Omega \sum_J \sum_{j=1}^{n+1} \left| \frac{\partial f_J}{\partial \overline{z}_j} \right|^2 e^{-\phi} dz - \int_\Omega \sum_{J'} \sum_{j,k=1}^{n+1} \frac{\partial f_{jJ'}}{\partial \overline{z}_k} \frac{\partial \overline{f_{kJ'}}}{\partial z_j} e^{-\phi} dz.$$

Next by (7),

$$\|\overline{\partial}_\phi^* f\|_\phi^2 = \int_\Omega \sum_{J'} \sum_{j,k=1}^{n+1} \frac{\partial (f_{jJ'} e^{-\phi})}{\partial z_j} \frac{\partial (\overline{f_{kJ'}} e^{-\phi})}{\partial \overline{z}_k} e^{\phi} dz$$

The leading term of this formula (where the derivatives do not hit $e^{-\phi}$) looks so much like the second term of the formula we had for $\|\overline{\partial} f\|_\phi^2$, except that the derivatives $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial \overline{z}_k}$ are swapped. This suggests that we should integrate by parts in $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial \overline{z}_k}$. We first integrate by parts in $\frac{\partial}{\partial z_j}$ and get

$$\begin{aligned}
\|\overline{\partial}_\phi^* f\|_\phi^2 = &- \int_\Omega \sum_{J'} \sum_{j,k=1}^{n+1} f_{jJ'} \frac{\partial^2 (\overline{f_{kJ'}} e^{-\phi})}{\partial z_j \partial \overline{z}_k} dz - \int_\Omega \sum_{J'} \sum_{j,k=1}^{n+1} f_{jJ'} \frac{\partial (\overline{f_{kJ'}} e^{-\phi})}{\partial \overline{z}_k} \frac{\partial \phi}{\partial z_j} dz \\
&+ \int_{\partial\Omega} \sum_{J'} \sum_{j,k=1}^{n+1} f_{jJ'} \frac{\partial (\overline{f_{kJ'}} e^{-\phi})}{\partial \overline{z}_k} \frac{\partial \rho}{\partial z_j} d\sigma
\end{aligned}$$

The boundary term vanishes by (6) since $f \in \mathrm{Dom}(\overline{\partial}_\phi^*)$. Next we integrate by parts in $\frac{\partial}{\partial \overline{z}_k}$:

$$\begin{aligned}
\|\overline{\partial}_\phi^* f\|_\phi^2 = &\int_\Omega \sum_{J'} \sum_{j,k=1}^{n+1} \frac{\partial f_{jJ'}}{\partial \overline{z}_k} \frac{\partial (\overline{f_{kJ'}} e^{-\phi})}{\partial z_j} dz - \int_{\partial\Omega} \sum_{J'} \sum_{j,k=1}^{n+1} f_{jJ'} \frac{\partial (\overline{f_{kJ'}} e^{-\phi})}{\partial z_j} \frac{\partial \rho}{\partial \overline{z}_k} d\sigma \\
&+ \int_\Omega \sum_{J'} \sum_{j,k=1}^{n+1} \frac{\partial f_{jJ'}}{\partial \overline{z}_k} \overline{f_{kJ'}} \frac{\partial \phi}{\partial z_j} e^{-\phi} dz + \int_\Omega \sum_{J'} \sum_{j,k=1}^{n+1} f_{jJ'} \overline{f_{kJ'}} \frac{\partial^2 \phi}{\partial z_j \partial \overline{z}_k} e^{-\phi} dz \\
&+ \int_{\partial\Omega} \sum_{J'} \sum_{j,k=1}^{n+1} f_{jJ'} \overline{f_{kJ'}} \frac{\partial \phi}{\partial z_j} \frac{\partial \rho}{\partial \overline{z}_k} e^{-\phi} d\sigma \\
= &\int_\Omega \sum_{J'} \sum_{j,k=1}^{n+1} \frac{\partial f_{jJ'}}{\partial \overline{z}_k} \frac{\partial \overline{f_{kJ'}}}{\partial z_j} e^{-\phi} dz + \int_\Omega \sum_{J'} \sum_{j,k=1}^{n+1} f_{jJ'} \overline{f_{kJ'}} \frac{\partial^2 \phi}{\partial z_j \partial \overline{z}_k} e^{-\phi} dz \\
&- \int_{\partial\Omega} \sum_{J'} \sum_{j,k=1}^{n+1} f_{jJ'} \frac{\partial \overline{f_{kJ'}}}{\partial z_j} \frac{\partial \rho}{\partial \overline{z}_k} e^{-\phi} d\sigma
\end{aligned}$$

But now since $f \in \mathrm{Dom}(\overline{\partial}_\phi^*)$, by (6) again, the vector field

$$\sum_{j,k=1}^{n+1} f_{jJ'} \frac{\partial}{\partial z_j}$$

is always tangent to $\partial\Omega$. Hence

$$\sum_{j=1}^{n+1} f_{jJ'} \frac{\partial}{\partial z_j} \left(\sum_{k=1}^{n+1} \overline{f_{kJ'}} \frac{\partial \rho}{\partial \overline{z}_k} \right) = 0 \quad \text{on } \partial\Omega.$$

In particular,

$$\sum_{j,k=1}^{n+1} f_{jJ'} \frac{\partial \overline{f_{kJ'}}}{\partial z_j} \frac{\partial \rho}{\partial \overline{z}_k} = - \sum_{j,k=1}^{n+1} f_{jJ'} \overline{f_{kJ'}} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} \quad \text{on } \partial\Omega.$$

Hence

$$(8)\qquad \|\overline{\partial} f\|_\phi^2 + \|\overline{\partial}_\phi^* f\|_\phi^2 = \int_\Omega \sum_J \sum_{j=1}^{n+1} \left|\frac{\partial f_J}{\partial \overline{z}_j}\right|^2 e^{-\phi} dz + \int_\Omega \sum_{J'} \sum_{j,k=1}^{n+1} f_{jJ'} \overline{f_{kJ'}} \frac{\partial^2 \phi}{\partial z_j \partial \overline{z}_k} e^{-\phi} dz$$
$$+ \int_{\partial\Omega} \sum_{J'} \sum_{j,k=1}^{n+1} f_{jJ'} \overline{f_{kJ'}} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} e^{-\phi} d\sigma.$$

The first term on the right hand side is certainly non-negative, and if we remember that Ω is pseudoconvex and that $\sum_{j=1}^{n+1} f_{jJ'} \frac{\partial}{\partial z_j} \in T_z^{(1,0)}(\partial\Omega)$, we see that the boundary integral is non-negative as well. If we now take $\phi = |z|^2$, we get the desired a priori weighted estimate (5) because with this ϕ,

$$\frac{\partial^2 \phi}{\partial z_j \partial \overline{z}_k} = \delta_{jk},$$

and the second term on the right hand side of the previous equation becomes $\|f\|_\phi^2$.

Note how pseudoconvexity enters crucially into the proof of this basic estimate. Note also incidentally that in the above basic estimate, the $\frac{\partial}{\partial \overline{z}_j}$ derivatives of all the components of f are automatically controlled by $\|\overline{\partial} f\|_\phi^2 + \|\overline{\partial}_\phi^* f\|_\phi^2$. This is not the case for the $\frac{\partial}{\partial z_j}$ derivatives; thus these two kinds of derivatives play a fundamentally different role in any deeper analysis of the inhomogeneous Cauchy-Riemann equation.

Another remark is that while it is relatively easy to choose the weight ϕ in our current setting, if we carry out an analogous analysis on a domain embedded in a *complex manifold* rather than $\mathbb{C}^{n+1}$, it will not be as clear what weight ϕ we should pick. The correct setting is then to work on a special class of complex manifolds called *Stein manifolds*, named after Karl Stein.

To sum up, assuming one knows how to pass to the real estimate using a density argument, we have shown the existence of weak solution in $L^2_{(0,q)}(\Omega)$ of the inhomogeneous Cauchy-Riemann equation (1) whenever Ω is bounded pseudoconvex with smooth boundary, and whenever $f \in L^2_{(0,q+1)}(\Omega)$ satisfies $\overline{\partial} f = 0$.

2. The $\overline{\partial}$-Neumann problem

There is another important partial differential equation in several complex variables that is closely related to the inhomogeneous Cauchy-Riemann equation. It is called the *$\overline{\partial}$-Neumann problem*, and it is such called because, as we shall see in the next Section, the most difficult component of this system of equations involves a boundary condition that is given by a complex normal derivative. To formulate the problem, we define an (unbounded) linear operator

$$\Box \colon L^2_{(0,q)}(\Omega) \to L^2_{(0,q)}(\Omega)$$

with domain

$$\mathrm{Dom}(\Box) := \{u \in L^2_{(0,q)}(\Omega) \colon u \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*), \overline{\partial} u \in \mathrm{Dom}(\overline{\partial}^*), \overline{\partial}^* u \in \mathrm{Dom}(\overline{\partial})\}$$

which is clearly dense in $L^2_{(0,q)}(\Omega)$. (From now on, unless otherwise specified, we shall again just put the standard Euclidean inner product on $L^2_{(0,q)}(\Omega)$.) For $u \in \mathrm{Dom}(\Box)$, we define

$$\Box u := (\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial})u.$$

This is usually called the *Kohn Laplacian.* One can check that it is an unbounded self-adjoint operator on $L^2_{(0,q)}(\Omega)$ for all $q \geq 0$. The *$\overline{\partial}$-Neumann problem* is to solve for $u \in \mathrm{Dom}(\Box)$ such that

$$\Box u = f \tag{9}$$

given $f \in L^2_{(0,q)}(\Omega)$.

For this equation to have a solution, again we may need some compatibility conditions on f. Since $\Box$ is self-adjoint, we have an orthogonal direct sum

$$L^2_{(0,q)}(\Omega) = \mathrm{Kernel}(\Box) \oplus \overline{\mathrm{Range}(\Box)}$$

for all $q \geq 0$. Hence for the equation (9) to have a solution, i.e. for $f \in L^2_{(0,q)}(\Omega)$ to be in $\mathrm{Range}(\Box)$, we need at least f to be orthogonal to the kernel of $\Box$ on $L^2_{(0,q)}(\Omega)$. The kernel of $\Box$, however, is easily checked to be just $\mathrm{Kernel}(\overline{\partial}) \cap \mathrm{Kernel}(\overline{\partial}^*)$, and we have already seen from our analysis of the inhomogeneous Cauchy-Riemann equation that this is trivial on $L^2_{(0,q)}(\Omega)$ when $q \geq 1$. When $q = 0$, the kernel of $\Box$ is just the kernel of $\overline{\partial}$, i.e. the space of holomorphic functions on Ω. Hence to solve (9), we shall require f to be orthogonal to holomorphic functions when $q = 0$, but we shall not require any orthogonality condition on f when $q \geq 1$.

In fact our solution to the inhomogeneous Cauchy-Riemann equation already allows us to solve the $\overline{\partial}$-Neumann problem weakly on bounded pseudoconvex domains with smooth boundaries. From the above analysis, to solve the $\overline{\partial}$-Neumann problem amounts to showing that $\Box$ has closed range in L^2. Here, however, we shall take a more explicit approach, by giving an explicit formula for the solution operator to the $\overline{\partial}$-Neumann problem in terms of the relative solution operators of $\overline{\partial}$ and $\overline{\partial}^*$ (see below for the definitions of these). At this stage it is best to keep track of the levels of forms on which operators are acting by various subscripts[1]. We shall write

$$\overline{\partial}_q \colon L^2_{(0,q)}(\Omega) \to L^2_{(0,q+1)}(\Omega), \quad \overline{\partial}^*_q \colon L^2_{(0,q+1)}(\Omega) \to L^2_{(0,q)}(\Omega) \quad (0 \leq q \leq n)$$

and

$$\Box_q = \overline{\partial}_{q-1}\overline{\partial}^*_{q-1} + \overline{\partial}^*_q\overline{\partial}_q \colon L^2_{(0,q)}(\Omega) \to L^2_{(0,q)}(\Omega) \quad (0 \leq q \leq n+1).$$

We have seen that the ranges of $\overline{\partial}_q$ are always closed on such domains Ω, and hence so are the ranges of $\overline{\partial}^*_q$. It follows that for all $0 \leq q \leq n$,

$$L^2_{(0,q+1)}(\Omega) = \mathrm{Range}(\overline{\partial}_q) \oplus \mathrm{Kernel}(\overline{\partial}^*_q) \quad \text{and} \quad L^2_{(0,q)}(\Omega) = \mathrm{Range}(\overline{\partial}^*_q) \oplus \mathrm{Kernel}(\overline{\partial}_q). \tag{10}$$

It is also convenient to introduce the orthogonal projections B_q and B'_q onto the kernels of $\overline{\partial}_q$ and $\overline{\partial}^*_q$ respectively, namely

$$B_q \colon L^2_{(0,q)}(\Omega) \to \mathrm{Kernel}(\overline{\partial}_q), \qquad B'_q \colon L^2_{(0,q+1)}(\Omega) \to \mathrm{Kernel}(\overline{\partial}^*_q).$$

(B_0 is usually called the *Bergman projection*; it is just the orthogonal projection onto the closed subspace of holomorphic functions.) Then there exists bounded linear operators

$$K_q \colon L^2_{(0,q+1)}(\Omega) \to (\mathrm{Kernel}(\overline{\partial}_q))^{\perp} \cap \mathrm{Dom}(\overline{\partial}_q)$$

and

$$K'_q \colon L^2_{(0,q)}(\Omega) \to (\mathrm{Kernel}(\overline{\partial}^*_q))^{\perp} \cap \mathrm{Dom}(\overline{\partial}^*_q)$$

(here $\perp$ denotes orthogonal complement in the respective L^2 spaces) such that

$$\overline{\partial}_q K_q = I - B'_q \quad \text{on } L^2_{(0,q+1)}(\Omega)$$

[1]Here the notations are chosen such that $\overline{\partial}_q$ acts on $(0,q)$ forms, $\overline{\partial}^*_q$ is the adjoint of $\overline{\partial}_q$, K_q and K'_q solve $\overline{\partial}_q$ and $\overline{\partial}^*_q$ relatively, B_q and B'_q project onto the kernels of $\overline{\partial}_q$ and $\overline{\partial}^*_q$, $\Box_q$ acts on $(0,q)$ forms, and N_q solves $\Box_q$.

and

$$\overline{\partial}_q^* K_q' = I - B_q \quad \text{on } L^2_{(0,q)}(\Omega)$$

for all $0 \le q \le n$, where I is the identity operator on the appropriate level of forms. K_q and K_q' are usually called the *relative solution operators* of $\overline{\partial}_q$ and $\overline{\partial}_q^*$ respectively, and they are adjoints of each other (c.f. Exerise 12).

Now for $0 \le q \le n+1$, define a bounded linear operator

$$N_q \colon L^2_{(0,q)}(\Omega) \to L^2_{(0,q)}(\Omega)$$

by

$$N_q = K_{q-1}' K_{q-1} + K_q K_q' \tag{11}$$

where K_{-1}, K_{-1}', K_{n+1} and K_{n+1}' are defined to be zero. Then N_q maps into $\text{Dom}(\Box_q)$, and

$$\Box_q N_q = \begin{cases} I & \text{if } q \ge 1 \\ I - B_0 & \text{if } q = 0 \end{cases}. \tag{12}$$

(c.f. Exercise 15.) Thus for $q \ge 1$, $u := N_q f \in \text{Dom}(\Box_q)$ solves $\Box_q u = f$ weakly for all $f \in L^2_{(0,q)}(\Omega)$, and if $f \in L^2_{(0,0)}(\Omega)$ is orthogonal to holomorphic functions, then $u := N_0 f$ solves $\Box_0 u = f$ weakly. Of course all these are only true under our standing assumption that Ω is bounded, smooth and pseudoconvex.

The operator N_q is usually called the *$\overline{\partial}$-Neumann operator*. It is also self-adjoint like $\Box_q$ (c.f. Exercise 16). We can also define a corresponding operator when we put a weighted L^2 inner product on $L^2_{(0,q)}(\Omega)$, and as we shall see, this is useful in the study of the existence of *some* C^∞ solutions to $\overline{\partial} u = f$ *without* requiring orthogonality of the solution to the kernel of $\overline{\partial}$ in any sense.

3. Regularity theory

In the previous Section we have seen how the solution to the inhomogeneous Cauchy-Riemann equation leads to a solution of the $\overline{\partial}$-Neumann problem. On the other hand, it is also well-known that this process can be reversed. In fact if $\Box_q$ has closed range in L^2, then both $\overline{\partial}_q$ and $\overline{\partial}_{q-1}$ have closed ranges in L^2 (c.f. Exercise 18), whereas we have essentially shown the converse of this in the previous Section. When the domain is bounded, smooth and pseudoconvex, we also have the following solution formula

$$K_q = \overline{\partial}_q^* N_{q+1} \quad \text{and} \quad K_q' = \overline{\partial}_q N_q \tag{13}$$

for all $0 \le q \le n$, expressing the relative solution operators of $\overline{\partial}$ and $\overline{\partial}^*$ in terms of the $\overline{\partial}$-Neumann operator (c.f. Exercise 19). These formula basically follow from the fact that

$$\overline{\partial}_{q-1}\overline{\partial}_{q-1}^* N_q + \overline{\partial}_q^* \overline{\partial}_q N_q = \begin{cases} I & \text{if } q \ge 1 \\ I - B_0 & \text{if } q = 0, \end{cases}$$

and the essence of these formula can be summarized in the following orthogonal decomposition

$$L^2_{(0,q)}(\Omega) = \overline{\partial}\overline{\partial}^* \text{Dom}(\Box) \oplus \overline{\partial}^* \overline{\partial} \text{Dom}(\Box) \oplus \text{Kernel}(\Box)$$

when $0 \le q \le n+1$ (with the kernel of $\Box$ being trivial when $q \ge 1$).

In fact the $\overline{\partial}$-Neumann problem is usually *easier* to study than the inhomogeneous Cauchy-Riemann equation. This is because, as we shall see shortly, that the $\overline{\partial}$-Neumann problem is a *boundary value problem*, and we have a whole host of tools to tackle such, such as pseudodifferential operators and Poisson operators. On the other hand, the orthogonality condition $u \perp \text{Kernel}(\overline{\partial})$ is rather difficult to deal with in general, because for instance any microlocalization or multiplication by cut-offs kills this property. It should be noted that when we introduced $\overline{\partial}^*$ into the study of the inhomogeneous Cauchy-Riemann equation, we were already in essence getting rid of this orthogonality condition by considerations similar to the $\overline{\partial}$-Neumann problem.

From (13), we already see that the regularity of the $\overline{\partial}$-Neumann operator N_{q+1} implies some regularity of the relative fundamental solutions K_q of $\overline{\partial}_q$. It is the regularity of N_q, $q \geq 1$, to which we now turn.

3.1. L^2 **Sobolev regularity.** We shall discuss two aspects of the regularity of N_q, $q \geq 1$, namely regularity in the Sobolev spaces H^k (the space of functions with k weak derivatives in L^2) and sharp regularity in L^p. We shall be rather brief in the former, and content ourselves to stating a few results with only a very brief indication of proofs.

Again let Ω be a bounded pseudoconvex domain with smooth boundary in $\mathbb{C}^{n+1}$, $n \geq 1$. We write $H^k_{(0,q)}(\Omega)$ for the space of $(0,q)$ forms on Ω with H^k coefficients. Our first result makes use of the weighted inner product (4) and the corresponding $\overline{\partial}$-Neumann operator. We shall write $\overline{\partial}_t^*$ for the adjoint of $\overline{\partial}$ under the weighted inner product with weight $\phi(z) := t|z|^2$, N_t for the corresponding $\overline{\partial}$-Neumann operator, and B_t be the corresponding Bergman projection. (Note however that the H^k Sobolev spaces we use were defined using the unweighted inner product.) The fisrt result is then

Theorem 1 (Exact regularity in H^k)**.** *For every $k \in \mathbb{N}$, there exists $T_k > 0$ such that*

(i) N_t, $\overline{\partial}_t^* N_t$, $\overline{\partial}\overline{\partial}_t^* N_t$ maps $H^k_{(0,q)}(\Omega)$ boundedly into itself whenever $t > T_k$, $q \geq 1$;
(ii) B_t maps $H^k_{(0,0)}(\Omega)$ boundedly into itself whenever $t > T_k$.

In particular, for every $k \in \mathbb{N}$ and every $f \in H^k_{(0,q+1)}(\Omega)$ with $\overline{\partial} f = 0$, $q \geq 0$, there exists a solution u to the equation $\overline{\partial} u = f$ with $u \in H^k_{(0,q)}(\Omega)$, because then one can just take $u = \overline{\partial}_t^* N_t f$.

We shall not give the detailed proof of this theorem, except to mention that the main thrust of the theorem is in bounding k *tangential derivatives* of the corresponding operators, and this can be done using the basic estimate and commuting derivatives. For instance, if say f is localized to a coordinate patch and T^k are k tangential derivatives, then the basic estimate implies

$$t\|T^k N_t f\|^2_{t|z|^2} \leq C(\overline{\partial} T^k N_t f, \overline{\partial} T^k N_t f)_{t|z|^2} + C(\overline{\partial}_t^* T^k N_t f, \overline{\partial}_t^* T^k N_t f)_{t|z|^2}.$$

If one integrate by parts and commute derivatives to let the $\overline{\partial}$ and $\overline{\partial}_t^*$ operators fall on N_t, then one can obtain a good estimate of the right hand side plus some error terms, and the errors can then be absorbed into the left hand side as long as t is sufficiently big.

Note that $\overline{\partial}\overline{\partial}_t^* N_t$ and B_t in the theorem are just (weighted) L^2 orthogonal projections onto the kernel of $\overline{\partial}$ on the appropriate levels of forms. Using their regularity as stated above, we get as a consequence

Theorem 2 (Existence of classical solutions)**.** *For every $k \in \mathbb{N}$, $q \geq 0$ and every $(0, q+1)$ form f on Ω that is C^∞ up to the boundary and satisfies $\overline{\partial} f = 0$, there exists a solution u to the equation $\overline{\partial} u = f$ that is C^∞ up to the boundary.*

This is because for each k, we can get, using the first theorem, a solution u_k to $\overline{\partial} u = f$ with $u_k \in H^k$. One is tempted to take limit as k goes to infinity to obtain a classical solution, but this doesn't quite work because the u_k's are defined only up to the kernel of $\overline{\partial}$, and there is no guarantee that these are close as $k \to \infty$. Nevertheless, if we correct each of these u_k by a suitable element in the kernel of $\overline{\partial}$ that is sufficiently regular, one can then pass to limits in all the H^k spaces. This can be done using the regularity of the projection operators onto the kernel of $\overline{\partial}$ that we described above, and would give a proof of the theorem. Again we omit the details.

One last aspect of Sobolev regularity that we mention here is that as long as Ω is *strongly* pseudoconvex (or satisfy some suitable kind of *finite type* conditions, which intuitively says that it is not entirely flat on the boundary), then one actually begins to *gain* derivatives; for instance, if Ω is a bounded strongly pseudoconvex domain with smooth boundary, then the (unweighted) $\overline{\partial}$-Neumann operator N satisfies

$$\|Nf\|_{H^{k+1}} \leq C\|f\|_{H^k}$$

for all $f \in H^k_{(0,q)}(\Omega)$, $q \geq 1$, $k \geq 0$. Note that while N solves a second order partial differential equation (namely $\Box u = f$), it cannot in general gain more than 1 derivative. This is because of the *subelliptic* nature of the problem; we shall discuss this further when we discuss the sharp L^p regularity of N. In fact in these problems whether or not one gains derivatives and how many derivatives one gain is usually related to the geometry of the boundary in a very delicate way. The above gain in N has its origin in the following improvement of our basic estimate when the domain is *strongly* pseudoconvex, namely

$$\|f\|^2_{H^{1/2}} \leq C(\|\overline{\partial} f\|^2_{L^2} + \|\overline{\partial}^* f\|^2_{L^2})$$

for $f \in L^2_{(0,q)}(\Omega) \cap \text{Dom}(\overline{\partial}) \cap \text{Dom}(\overline{\partial}^*)$, $q \geq 1$. This is usually called the 1/2-subelliptic estimate for bounded strongly pseudoconvex domains, and can be deduced from the basic identity (8).

3.2. **A model case: the upper half space.** Now we turn to study sharp L^p regularity of the $\overline{\partial}$-Neumann operator N_q; again $q \geq 1$. For simplicity, we shall limit ourselves to studying the $\overline{\partial}$-Neumann operator on a model space, namely the upper half-space

$$\mathcal{U}^{n+1} := \{w = (w', w_{n+1}) \in \mathbb{C}^{n+1} \colon \text{Im } w_{n+1} > |w'|^2\}.$$

This is a natural domain to study because it is biholomorphic to the unit ball in $\mathbb{C}^{n+1}$, and is easily checked to be *strongly* pseudoconvex. In fact this is a prototype of a strongly pseudoconvex domain, and our analysis that follows can also be carried over to all bounded strongly pseudoconvex domains with smooth boundaries with an additional effort to take care of the technicalities that arise. (Our methods, and the results that we shall prove, fail for domains that are just pseudoconvex though.) We shall basically give an explicit solution formula for the $\overline{\partial}$-Neumann operator in terms of operators that are more tractable, and in particular in terms of operators whose regularity properties are well-understood. The solution formula is sometimes also called a *parametrix* for the Kohn Laplacian.

Note that since $\mathcal{U}^{n+1}$ is not compact, we do not expect to get a solution to $\Box u = f$ in L^2 when $f \in L^2$; we shall thus just content ourselves to proving a priori estimates to the solution of this equation. In other words, we derive estimates of $u \in \text{Dom}(\Box)$ assuming that u is smooth up to the boundary, compactly supported in $\overline{\mathcal{U}^{n+1}}$, and solves $\Box u = f$ classically.

To begin with, let us describe in more explicit terms the Kohn Laplacian on $\mathcal{U}^{n+1}$. A basis of $(0,1)$ forms on $\mathcal{U}^{n+1}$ can be given by

$$\overline{\omega}_j := d\overline{w}_j \quad (1 \leq j \leq n) \quad \text{and} \quad \overline{\omega}_{n+1} := 2^{1/2}\overline{\partial}\rho = i2^{-1/2} d\overline{w}_{n+1} - 2^{1/2}\sum_{j=1}^{n} \overline{w}_j d\overline{w}_j,$$

where $\rho := \text{Im } w_{n+1} - |w'|^2$ is (the negative of) a defining function for $\mathcal{U}^{n+1}$, and the corresponding dual basis of anti-holomorphic vector fields is then given by

$$\overline{Z}_j := \frac{\partial}{\partial \overline{w}_j} - 2iw_j \frac{\partial}{\partial \overline{w}_{n+1}} \quad (1 \leq j \leq n) \quad \text{and} \quad \overline{Z}_{n+1} := -i2^{1/2}\frac{\partial}{\partial \overline{w}_{n+1}}.$$

The Euclidean coordinates w, however, is not the best coordinate system on $\mathcal{U}^{n+1}$ in which we can describe these $(0,1)$ forms and anti-holomorphic vector fields; hence we shall introduce another coordinate system $[z,t,\rho]$ where the defining function ρ plays a more visible role. Let now

$$z = w', \quad t = \text{Re } w_{n+1}, \quad \rho = \text{Im } w_{n+1} - |w'|^2$$

so that $\mathcal{U}^{n+1} = \{[z,t,\rho] \in \mathbb{C}^n \times \mathbb{R} \times \mathbb{R} \colon \rho > 0\}$. This is not a holomorphic change of coordinates; in particular $t + i\rho$ is not a holomorphic function on $\mathcal{U}^{n+1}$. Hence in this coordinate system the anti-holomorphic vector fields is no longer spanned by the $\frac{\partial}{\partial \overline{z}_j}$'s; instead the previous basis of anti-holomorphic vector fields is now written

$$\overline{Z}_j = \frac{\partial}{\partial \overline{z}_j} - iz_j\frac{\partial}{\partial t} \quad (1 \leq j \leq n) \quad \text{and} \quad \overline{Z}_{n+1} = -i2^{-1/2}\left(\frac{\partial}{\partial t} + i\frac{\partial}{\partial \rho}\right),$$

and the previous basis of $(0,1)$ forms is written $\overline{\omega}_j = dz_j$ $(1 \leq j \leq n)$, $\overline{\omega}_{n+1} = 2^{1/2}\overline{\partial}\rho$.

There are three advantages of using the basis $\{\overline{\omega}_j\}$ and $\{\overline{Z}_j\}$. First the first n vector fields above are tangent to the boundary $\partial\mathcal{U}^{n+1}$ of $\mathcal{U}^{n+1}$, because by definition $\overline{Z}_j\rho = 0$ for $j = 1, \dots, n$. Second the $\overline{Z}_j$'s, $1 \leq j \leq n+1$ all commute with each other, i.e.

$$[\overline{Z}_j, \overline{Z}_k] = 0 \quad \text{for all } 1 \leq j,k \leq n+1,$$

because

$$\overline{\partial}\overline{\omega}_j = 0 \quad \text{for all } 1 \leq j \leq n+1. \tag{14}$$

Finally if we write Z_j for the complex conjugate of $\overline{Z}_j$, i.e.

$$Z_j = \frac{\partial}{\partial z_j} + i\overline{z}_j\frac{\partial}{\partial t} \quad (1 \leq j \leq n) \quad \text{and} \quad Z_{n+1} = i2^{-1/2}\left(\frac{\partial}{\partial t} - i\frac{\partial}{\partial \rho}\right),$$

then Z_j and $\overline{Z}_k$ obey a simple commutation relation, namely

$$[Z_j, \overline{Z}_k] = \begin{cases} -2iT & \text{if } 1 \leq j,k \leq n \text{ and } j = k \\ 0 & \text{otherwise,} \end{cases} \tag{15}$$

where

$$T := \frac{\partial}{\partial t}.$$

The fact that Z_j commutes with all $\overline{Z}_k$ when $j \neq k$ shall greatly simplify our computations below.

We now define a Hermitian inner product $\langle \cdot, \cdot \rangle$ on the space of $(0,q)$ forms at a point so that the $\overline{\omega}_I$'s, when I ranges over all strictly increasing multi-indices of length q, form an orthonormal basis at each point; here $\overline{\omega}_I := \overline{\omega}_{i_1} \wedge \cdots \wedge \overline{\omega}_{i_q}$ if $I = (i_1, \dots, i_q)$. This then induces a Hermitian inner product on the space $L^2_{(0,q)}(\mathcal{U}^{n+1})$ of $(0,q)$ forms with L^2 coefficients, namely

$$(u,v) = \int_{\{\rho>0\}} \langle u, v\rangle dz dt d\rho$$

which makes $L^2_{(0,q)}(\mathcal{U}^{n+1})$ a Hilbert space. (Note that the change of coordinates carries the Euclidean measure dw in the old coordinate system to the Euclidean measure $dzdtd\rho$ in the new coordinate system.)

The distributional $\overline{\partial}$ is now given by the formula

$$\overline{\partial} u = \sum_I \sum_{j=1}^{n+1} \overline{Z}_j(u_I)\overline{\omega}_j \wedge \overline{\omega}_I \quad \text{if } u = \sum_I u_I \overline{\omega}_I;$$

this just follows from our alternative definition of $\overline{\partial}$ in Section 1, and (14). We can then define the Hilbert space operators $\overline{\partial}$, $\overline{\partial}^*$ and $\Box$; explicitly, if a $(0,q)$ form $u = \sum_I u_I \overline{\omega}_I$ is smooth up to boundary and has compact support on $\overline{\mathcal{U}^{n+1}}$, then

$$u \in \mathrm{Dom}(\overline{\partial}^*) \quad \text{if and only if} \quad u_I = 0 \quad \text{on } \partial U^{n+1} \text{ whenever } n+1 \in I$$

in which case

$$\overline{\partial}^* u = \sum_{J'} \sum_{j=1}^{n+1} Z_j(u_{jJ'}) d\overline{z}_{J'},$$

whereas $u \in \mathrm{Dom}(\Box)$ if and only if on $\partial\mathcal{U}^{n+1}$,

$$\begin{cases} u_I = 0 & \text{whenever } n+1 \in I \\ \overline{Z}_{n+1} u_I = 0 & \text{whenever } n+1 \notin I, \end{cases}$$

in which case

$$\Box u = \sum_{n+1 \notin I} \Box^\tau(u_I)\overline{\omega}_I + \sum_{n+1 \in I} \Box^\nu(u_I)\overline{\omega}_I$$

where $\Box^\tau$, $\Box^\nu$ are scalar differential operators acting on functions, defined by

$$\Box^\tau = \Box^\tau_q := \mathcal{L}_{n-2q} - \frac{1}{2}\left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial \rho^2}\right)$$

and

$$\Box^\nu = \Box^\nu_q := \mathcal{L}_{n-2q+2} - \frac{1}{2}\left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial \rho^2}\right)$$

Here the $\mathcal{L}_\alpha$'s are scalar tangential differential operators acting on functions, given by

$$\mathcal{L}_\alpha := -\frac{1}{2}\sum_{j=1}^n \left(Z_j\overline{Z}_j + \overline{Z}_j Z_j\right) + i\alpha T = -\frac{1}{2}\sum_{j=1}^n \left(X_j^2 + Y_j^2\right) + i\alpha T$$

where X_j and Y_j are the real vector fields defined by

$$\overline{Z}_j = \frac{1}{2}(X_j + iY_j).$$

These formula can be checked using the definitions and (15). Note that $\Box$ acts *component-wise*, and this nice decoupling of this system of equation only happens because Z_j commutes with $\overline{Z}_k$ for all $j \neq k$. This is special to the upper half-space $\mathcal{U}^{n+1}$ (and not true even for general strongly pseudoconvex domains), which makes it a simple model case to study.

From the above discussion we see that to study estimates for classical solutions to the $\overline{\partial}$-Neumann problem for $(0,q)$ forms on $\mathcal{U}^{n+1}$, it amounts to studying estimates to the following two boundary value problems for *scalar* functions ϕ:

$$\Box^\tau_q \phi = \psi \quad \text{on } \mathcal{U}^{n+1}, \qquad \overline{Z}_{n+1}\phi = 0 \quad \text{on } \partial\mathcal{U}^{n+1} \tag{16}$$

and

$$\Box_q^\nu \phi = \psi \quad \text{on } \mathcal{U}^{n+1}, \qquad \phi = 0 \quad \text{on } \partial\mathcal{U}^{n+1}. \tag{17}$$

The latter is an *elliptic* boundary value problem, and can be treated by classical theory. The former is more difficult because the boundary condition there involves a *complex* normal derivative $\overline{Z}_{n+1}$, and as we shall see this is not an elliptic boundary condition. As a result, the deepest analysis of the $\overline{\partial}$-Neumann problem involves studying this complex Neumann condition, and it is how the $\overline{\partial}$-Neumann problem got its name.

3.3. **Reduction to the boundary.** We now focus on the study of the tangential component (16) of the $\overline{\partial}$-Neumann equation. To do so, we follow a well-known paradiagm for studying general boundary value problem. This consists of a reduction to a study of (pseudo)differential operators the boundary, but before that we need some preliminaries.

First, an important fact about $\mathcal{U}^{n+1}$ here is that its boundary $\partial\mathcal{U}^{n+1}$ happens to carry a *group* structure; if we parametrize that by assigning a point $[z,t,0] \in \partial\mathcal{U}^{n+1}$ the coordinates $[z,t]$, then the group law is given by

$$[z,t][\zeta,s] := [z+\zeta, t+s+2\text{Im } (z\cdot\overline{\zeta})].$$

This is a non-abelian Lie group; it's usually called the Heisenberg group $\mathbb{H}^n$. The identity is $[0,0]$, and the inverse of $[z,t]$ is $[-z,-t]$. It acts on $\mathcal{U}^{n+1}$ by *translations*:

$$[\zeta,s][z,t,\rho] := [\zeta+z, s+t+2\text{Im } (\zeta\cdot\overline{z}), \rho].$$

In other words, it acts by leaving ρ fixed and operating on the $[z,t]$ variable by the group law of $\mathbb{H}^n$. Since $\mathbb{H}^n$ is a Lie group, we can define convolutions using *left* translations: we define

$$(\phi * F)([z,t]) := \int_{\mathbb{H}^n} \phi([\zeta,s])F([-\zeta,-s][z,t])d\zeta ds.$$

On $\mathbb{H}^n$ we have the restrictions of the tangential differential operators Z_j, $\overline{Z}_j$ $(1\leq j\leq n)$ and T; we also have the the restrictions of the tangential real vector fields X_j, Y_j and the tangential operator $\mathcal{L}_\alpha$. By abuse of notations we use the same symbol for the original operators and their restrictions. These restricted operators are now *left-invariant* under the group law of the Heisenberg group; in other words, they commute with left-translation operators $L_{[\zeta,s]}$, defined by

$$(L_{[\zeta,s]}f)[z,t] := f([\zeta,s][z,t]).$$

The important fact here is that the operator $\mathcal{L}_\alpha$ on $\mathbb{H}^n$ can then be solved by convoluting against a fundamental solution F_α when $\alpha \notin \{\pm(n+2m)\colon m\in\mathbb{Z}, m\geq 0\}$; in fact if

$$F_\alpha([z,t]) := \gamma_\alpha^{-1}(|z|^2-it)^{-\frac{n+\alpha}{2}}(|z|^2+it)^{-\frac{n-\alpha}{2}}, \quad \gamma_\alpha := \frac{2^{2-n}\pi^{n+1}}{\Gamma(\frac{n+\alpha}{2})\Gamma(\frac{n-\alpha}{2})},$$

then

$$\mathcal{L}_\alpha(\phi * F_\alpha) = \phi$$

for either Schwartz ϕ, or distributions ϕ with compact support, when α is not one of the above forbidden values. (Note that γ_α is finite exactly when α is not a forbidden value.)

We now return to the study of (16). Suppose ϕ is a function smooth up to the boundary, supported in a fixed compact set in $\overline{\mathcal{U}^{n+1}}$, and solves $\Box_q^\tau\phi=\psi$, $\overline{Z}_{n+1}\phi\big|_{\partial\mathcal{U}^{n+1}}=0$ classically. We shall construct a solution formula that recovers ϕ from ψ. First it was known that there is an explicit Green's operator G such that $G\psi$ satisfy

$$\Box_q^\tau(G\psi) = \psi \quad \text{on } \mathcal{U}^{n+1} \qquad \text{and} \quad G\psi = 0 \quad \text{on } \partial\mathcal{U}^{n+1}.$$

It follows that

$$\Box_q^\tau(\phi - G\psi) = 0 \quad \text{on } \mathcal{U}^{n+1},$$

and of course

$$\phi - G\psi = \phi_b \quad \text{on } \partial\mathcal{U}^{n+1}$$

where ϕ_b is the restriction of ϕ to the boundary $\partial\mathcal{U}^{n+1}$. Now taking normal derivative of the Green's operator, we get an explicit Poisson operator P, which solves

$$\Box_q^\tau(Ph_b) = 0 \quad \text{on } \mathcal{U}^{n+1} \qquad \text{and} \quad Ph_b = h_b \quad \text{on } \partial\mathcal{U}^{n+1}$$

for all continuous functions h_b defined on $\partial\mathcal{U}^{n+1}$. In particular, we get from the above

$$\phi = G\psi + P\phi_b. \tag{18}$$

Hence to solve for ϕ amounts to solving for ϕ_b. Now ϕ_b is determined by the boundary condition on ϕ; in fact, taking $\overline{Z}_{n+1}$ derivative of both sides and restricting to the boundary, we get, from our assumption that $\overline{Z}_{n+1}\phi = 0$ on $\partial\mathcal{U}^{n+1}$, that

$$\overline{Z}_{n+1}(P\phi_b)\big|_{\partial\mathcal{U}^{n+1}} = -\,\overline{Z}_{n+1}(G\psi)\big|_{\partial\mathcal{U}^{n+1}}.$$

Define the boundary operator $\Box_+$ by letting

$$\Box_+\phi_b = \overline{Z}_{n+1}(P\phi_b)\big|_{\partial\mathcal{U}^{n+1}};$$

it is a pseudodifferential operator on $\partial\mathcal{U}^{n+1}$. The condition for ϕ_b is now

$$\Box_+\phi_b = -\,\overline{Z}_{n+1}(G\psi)\big|_{\partial\mathcal{U}^{n+1}}.$$

If we could somehow invert $\Box_+$, then we could recover ϕ_b from the given data ψ, and we would have obtained a formula for ϕ in terms of ψ by (18). This is our next goal.

To do so we need to analyze $\Box_+$ more closely. Since $\overline{Z}_{n+1} = -i2^{-1/2}\left(T + i\frac{\partial}{\partial\rho}\right)$, T is tangential to the boundary and $P\phi_b = \phi_b$ on the boundary, we get

$$\Box_+\phi_b = -i2^{-1/2}(T + iN)\phi_b$$

where N is the *Dirichlet-to-Neumann operator*, given by

$$N\phi_b := \frac{\partial}{\partial\rho}(P\phi_b)\bigg|_{\rho=0}.$$

The trick now is that if we define

$$\Box_- := i2^{-1/2}(T - iN),$$

then

$$\Box_-\Box_+ = \frac{1}{2}(T^2 + N^2).$$

This is because T commutes with N. But N^2 is just

$$\frac{\partial^2}{\partial\rho^2}P_\rho\bigg|_{\rho=0}$$

where $(P_\rho h_b)([z,t]) := (Ph_b)([z,t,\rho])$, because $P_{\rho_1}P_{\rho_2} = P_{\rho_1+\rho_2}$. It follows that if we apply the previous equation to ϕ_b, then

$$\begin{aligned}\Box_-\Box_+\phi_b &= \frac{1}{2}\left(T^2 + \frac{\partial^2}{\partial\rho^2}\right)(P\phi_b)\bigg|_{\rho=0}\\ &= \mathcal{L}_{n-2q}(P\phi_b)\big|_{\rho=0}\\ &= \mathcal{L}_{n-2q}\phi_b.\end{aligned}$$

So if now $1 \le q \le n-1$, then from the fundamental solution F_{n-2q} to $\mathcal{L}_{n-2q}$ we get

$$\phi_b = \Box_-(\Box_+\phi_b) * F_{n-2q}$$

which allows us to invert $\Box_+$. It follows that for this range of q we have

$$\phi = G\psi - P\Box_-\left(\left(\overline{Z}_{n+1}G\psi\right)\big|_{\partial\mathcal{U}^{n+1}} * F_{n-2q}\right) \tag{19}$$

which is our desired parametrix for $\Box$.

A few remarks are in order. First the above solution formula only works when $q \neq n$. When $q = n$ the formula fails because $\mathcal{L}_{n-2q}$ then fails to be invertible. Nevertheless it is invertible modulo its null space, and a further microlocalization produces a (more complicated) solution formula. Note also that the case $q = n+1$ is irrelevant, because if we solve the $\overline{\partial}$-Neumann equation for $(0, n+1)$ form, then there is only one component, which is normal (hence the problem is actually elliptic), and the above analysis is not necessary.

Next, we remark that the boundary condition in (16) was said to be non-elliptic because the associated boundary operator $\Box_+$ is not elliptic as a pseudodifferential operator on $\partial\mathcal{U}^{n+1}$. This is because we are now using a *complex* normal derivative. If we had used just the real normal derivative, the boundary condition would then become elliptic and the analysis would become much easier.

Note, however, that we still say that the boundary condition in (16) is *subelliptic* because one still gains derivatives from that.

Finally, this solution formula is useful in studying the regularity of the solutions of the $\overline{\partial}$-Neumann problem on $\mathcal{U}^{n+1}$, because the regularity of each term is well-known. Without going into details of what exactly the regularities of these operators are, let us just state the following result on the sharp L^p regularity of the solutions of the $\overline{\partial}$-Neumann equation.

3.4. **Sharp L^p regularity.** Suppose $u = \sum_I u_I \overline{\omega}_I \in \text{Dom}(\Box)$ is a $(0,q)$ form $(q \geq 1)$ that is smooth up to the boundary, supported in a fixed compact set in $\overline{\mathcal{U}^{n+1}}$, and solves $\Box u = f$ classically. Then the normal components of u, namely u_I with $n+1 \in I$, gains two derivatives in L^p for all $1 < p < \infty$, i.e.

$$\|u_I\|_{\dot{W}^{2,p}} \leq C\|f_I\|_{L^p}.$$

This is because u_I then solves the elliptic boundary value problem (17). On the other hand, the tangential components of u, namely u_I with $n+1 \notin I$, satisfies

$$\|Q(Z,\overline{Z})u_I\|_{L^p} + \|\overline{Z}_{n+1}u_I\|_{\dot{W}^{1,p}} \leq C\|f_I\|_{L^p}$$

for all $1 < p < \infty$, where $Q(Z,\overline{Z})$ denotes any (non-commutative) purely quadratic polynomial of Z_j and $\overline{Z}_j$, $1 \leq j \leq n$. This involves a more detailed analysis of the operators that are involved in (19), which we refrain from giving, except to mention that two kinds of operators are involved, one that is more adapted to the Euclidean structure of the underlying space, and another that is more adapted to the non-commutative nature of the Heisenberg group.

The point here is that for the tangential components of the solutions of the $\overline{\partial}$-Neumann equation, one cannot expect to gain two full derivatives in every direction; rather, there is this subtle distinction between *good* directions (namely Z_j, $\overline{Z}_j$, $1 \leq j \leq n$ and $\overline{Z}_{n+1}$) and *bad* directions (namely Z_{n+1}). The best we can say about $Z_{n+1}u_I$ for $n+1 \notin I$ is just that it is in L^p, by writing it as a linear combination of $\overline{Z}_{n+1}u_I$ and $Tu_I = [Z_1, \overline{Z}_1]u_I$; it does not gain any further derivatives. Situations like this are typical in the analysis of subelliptic equations.

As an application of the above analysis, let us mention the following result about the inhomogeneous Cauchy-Riemann equation: if u is a $(0,q)$ form on $\mathcal{U}^{n+1}$ $(0 \leq q \leq n)$ that is smooth up to boundary, has support contained in a fixed compact subset of $\overline{\mathcal{U}^{n+1}}$ and is orthogonal to the kernel of $\overline{\partial}$, then

$$\sum_{j=1}^{n} \left(\|Z_j u\|_{L^p} + \|\overline{Z}_j u\|_{L^p} \right) + \|\overline{Z}_{n+1} u\|_{L^p} \leq C \|\overline{\partial} u\|_{L^p}$$

for $1 < p < \infty$. This can be proved using the formula $u = \overline{\partial}^* N(\overline{\partial} u)$ and analyzing the tangential and normal components of u separately.

4. Exercises

1. Verify that the two definitions of $\overline{\partial}$ given in Section 1 are equivalent.
2. Verify that $\overline{\partial}$ forms a complex.
3. Compute the Levi form of the unit ball $\{|z| < 1\}$ in $\mathbb{C}^{n+1}$. Is it (strongly) pseudoconvex?
4. Prove that every (smooth) convex domain is pseudoconvex, and strongly if the domain is strictly convex.
5. Show that a densely defined linear operator $T\colon H_1 \to H_2$ between two Hilbert spaces is closed if and only if the following is true: if $x_j \in \mathrm{Dom}(T)$, $x_j \to x$ and $Tx_j \to y$ then $x \in \mathrm{Dom}(T)$ and $Tx = y$.
6. Show that if $T\colon H_1 \to H_2$ is closed, then its Hilbert space adjoint $T^*\colon H_2 \to H_1$ is also densely defined, linear and closed, and $T^{**} = T$. (Hint: The graph of T^* is (up to a twist) the orthogonal complement of the graph of T in $H_1 \times H_2$.)
7. Show that if $T\colon H_1 \to H_2$ is closed, then the following are equivalent:
   (a) T has closed range in H_2;
   (b) $\|f\| \leq C\|Tf\|$ for all $f \in \mathrm{Dom}(T)$ orthogonal to the kernel of T;
   (c) $\|u\| \leq C\|T^* u\|$ for all $u \in \mathrm{Dom}(T^*)$ orthogonal to the kernel of T^*;
   (d) T^* has closed range in H_1.
   (Hint: To show (a) implies (b), use the closed graph theorem. (b) and (c) are equivalent by duality.)
8. Work out explicitly the weighted basic estimate for smooth $(0,1)$ forms when $\phi = 0$.
9. Show that if Ω is bounded, smooth and strongly pseudoconvex, then

   $$\|f\|^2_{L^2(\partial\Omega)} \leq C(\|\overline{\partial} f\|^2_{L^2} + \|\overline{\partial}^* f\|^2_{L^2})$$

   for all $(0,q)$ forms $f \in \mathrm{Dom}(\overline{\partial}^*)$ that are smooth up to boundary, where $q \geq 1$.

From now on assume that Ω is a bounded pseudoconvex domain with smooth boundary.

10. Write down a more explicit definition of the relative solution operators K_q and K'_q to $\overline{\partial}_q$ and $\overline{\partial}^*_q$ by considering the orthogonal decompositions (10).
11. Show that

    $$K_q \overline{\partial}_q = I - B_q \quad \text{on } \mathrm{Dom}(\overline{\partial}_q)$$

    and

    $$K'_q \overline{\partial}^*_q = I - B'_q \quad \text{on } \mathrm{Dom}(\overline{\partial}^*_q)$$

    for $0 \leq q \leq n$. (Hint: apply $\overline{\partial}$ and $\overline{\partial}^*$ to both sides respectively.)
12. Show that K_q and K'_q are adjoints of each other, for $0 \leq q \leq n$. (Hint: Show $\overline{\partial}_q (K'_q)^* = \overline{\partial}_q K_q$ where $(K'_q)^*$ is the adjoint of K'_q. To do so one can observe that $\overline{\partial}_q (K'_q)^* = (K'_q \overline{\partial}^*_q)^*$ and use the Exercise 11.)
13. Prove the unweighted basic estimate (5). (Hint: Suppose $q \geq 0$. Since we have already shown that $\mathrm{Range}(\overline{\partial}_q) = \mathrm{Kernel}(\overline{\partial}_{q+1})$, we have

    $$L^2_{(0,q+1)}(\Omega) = \mathrm{Kernel}(\overline{\partial}_{q+1}) \oplus \mathrm{Kernel}(\overline{\partial}^*_q),$$

    so

    $$B_{q+1} + B'_q = I.$$

For $f \in L^2_{(0,q+1)}(\Omega) \cap \text{Dom}(\overline{\partial}) \cap \text{Dom}(\overline{\partial}^*)$, $q \geq 0$, write

$$f = (I - B_{q+1})f + (I - B'_q)f.$$

But $I - B_{q+1} = (I - B_{q+1})^* = (K'_{q+1})^* \overline{\partial}_{q+1}$ and $(I - B'_q) = (I - B'_q)^* = K_q^* \overline{\partial}_q^*$ so

$$f = (K'_{q+1})^*(\overline{\partial}_{q+1} f) + K_q^*(\overline{\partial}_q^* f),$$

and one can then apply boundedness of K'_{q+1} and K_q on L^2.) Note what a long detour we have gone to prove this unweighted basic estimate: we had to first prove the weighted version, and draw some qualitative functional analytic consequences from that, before we could really prove the unweighted version.

14. Show that $\Box_q$ is self-adjoint on L^2 for $0 \leq q \leq n+1$. (Hint: Let $f \in \text{Dom}(\Box_q^*)$. For $h \in \text{Dom}(\overline{\partial}_{q-1})$,
$$(f, \overline{\partial}_{q-1} h) = (f, \overline{\partial}_{q-1} \overline{\partial}^*_{q-1} K'_{q-1} h) = (f, \Box_q K'_{q-1} h) = (K_{q-1} \Box_q^* f, h)$$
so $f \in \text{Dom}(\overline{\partial}^*_{q-1})$ and $\overline{\partial}^*_{q-1} f = K_{q-1} \Box_q^* f \in \text{Dom}(\overline{\partial}_{q-1})$. Similarly $f \in \text{Dom}(\overline{\partial}_q)$ etc, using the fact that $\overline{\partial}_q = (\overline{\partial}_q^*)^*$.)
15. Prove (12). (Hint: when $1 \leq q \leq n$,
$$\Box_q N_q = (I - B'_{q-1}) + (I - B_q) = I$$
because
$$\overline{\partial}^*_{q-1} K_q = 0 = \overline{\partial}_q K'_{q-1} \quad \text{and} \quad B_{q-1} K'_{q-1} = 0 = B'_q K'_q$$
and
$$B_q + B'_{q-1} = I.$$
A similar argument works when $q = n+1$ and $q = 0$. What makes $q = 0$ special in (12)?)
16. Show that N_q is self-adjoint on L^2 for $0 \leq q \leq n+1$. (Hint: use formula (11) for N_q.)
17. Show that $N_q \Box_q = \Box_q N_q$ on the domain of $\Box_q$ for $0 \leq q \leq n+1$. (Hint: use duality.)
18. Show that if
$$\Box_q \colon L^2_{(0,q)}(\Omega) \to L^2_{(0,q)}(\Omega)$$
has closed range, then both
$$\overline{\partial}_q \colon L^2_{(0,q)}(\Omega) \to L^2_{(0,q+1)}(\Omega) \quad \text{and} \quad \overline{\partial}_{q-1} \colon L^2_{(0,q-1)}(\Omega) \to L^2_{(0,q)}(\Omega)$$
have closed ranges.
19. Prove (13). Also prove that
$$\overline{\partial}_q^* N_{q+1} = N_q \overline{\partial}_q^* \quad \text{on Dom}(\overline{\partial}_q^*) \quad \text{and} \quad \overline{\partial}_q N_q = N_{q+1} \overline{\partial}_q \quad \text{on Dom}(\overline{\partial}_q)$$
for $0 \leq q \leq n$. Conclude again that K_q and K'_q are adjoints of each other, and that $N_q \Box_q = \Box_q N_q$ on the domain of $\Box_q$.

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